1 Controller Canonical Form - Introduction

a) Show that a discrete-time system in controllable canonical form is essentially a higher order scalar recurrence relation with scalar input.

Answer

Let our system be defined as follows:

\[

t(t + 1) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & \cdots & a_{n-1} & a_n
\end{bmatrix} t(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)
\]

\[\Rightarrow x_1(t + 1) = x_2(t)\]
\[x_2(t + 1) = x_3(t)\]
\[x_3(t + 1) = x_4(t)\]
\[\vdots\]
\[x_{n-1}(t + 1) = x_n(t)\]
\[x_n(t + 1) = \sum_{i=1}^{n} a_i x_i(t) + u(t)\]

To further simplify, let’s write \(x_1(t) = y(t)\). Hence,

\[x_1(t) = y(t)\]
\[x_2(t) = y(t + 1)\]
\[x_3(t) = y(t + 2)\]
\[\vdots\]
\[x_n(t) = y(t + n - 1)\]

With the above substitutions, we can write our discrete time system as an \(n^{th}\) order recurrence relation as follows:

\[x_n(t + 1) = y(t + n) = \sum_{i=1}^{n} a_i y(t + i - 1) + u(t)\]

b) Show that a continuous-time system in controllable canonical form is essentially a higher order scalar differential equation with scalar input.
Answer

Let our system be defined as follows:

$$\frac{d}{dt} \mathbf{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_{n-1} & a_n \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

Thus,

$$\frac{d}{dt} x_1(t) = x_2(t)$$
$$\frac{d}{dt} x_2(t) = x_3(t)$$
$$\frac{d}{dt} x_3(t) = x_4(t)$$
$$\vdots$$
$$\frac{d}{dt} x_{n-1}(t) = x_n(t)$$
$$\frac{d}{dt} x_n(t) = \sum_{i=1}^{n} a_i x_i(t) + u(t)$$

To further simplify, again let’s write $x_1(t) = y(t)$. Hence,

$$x_1(t) = y(t)$$
$$x_2(t) = \frac{d}{dt} y(t)$$
$$x_3(t) = \frac{d^2}{dt^2} y(t)$$
$$\vdots$$
$$x_n(t) = \frac{d^{n-1}}{dt^{n-1}} y(t)$$

With the above substitutions, we can write our continuous time system as an $n^{th}$ order differential equation as follows:

$$\frac{d}{dt} x_n(t) = \frac{d^n}{dt^n} y = \sum_{i=1}^{n} a_i \frac{d^{i-1}}{dt^{i-1}} y(t) + u(t)$$

Where, $\frac{d^n}{dt^n} \equiv 1$. Note the above differential equation will be homogenous if $u(t) = 0$ for all $t$
2 Controller Canonical Form - Eigenvalues Placement

Consider the following linear discrete time system

\[
\dot{x}(t + 1) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -3 & -4
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u(t)
\]

a) Is this system controllable?

**Answer**

We calculate

\[ C = [B, AB, A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \]

Observe that \( C \) matrix is full rank and hence our system is controllable.

b) Is the linear discrete time system stable?

**Answer**

Because the matrix \( A \) is in controllable canonical form, we can find the characteristic polynomial to be \( \lambda^3 + 4\lambda^2 + 3\lambda \). From that polynomial we calculate the eigenvalues of matrix \( A \):

\[
0 = \lambda^3 + 4\lambda^2 + 3\lambda = \lambda(\lambda + 3)(\lambda + 1)
\]

The eigenvalues are then 0, -3, -1. Since the eigenvalue at -3 is outside the unit circle, this system is unstable.

c) Using state feedback \( u(t) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \dot{x}(t) \) place the eigenvalues at 0, 1/2, -1/2.

**Answer**

The closed loop system is given by

\[
A + BK = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
k_1 & k_2 - 3 & k_3 - 4
\end{bmatrix}
\]

which has characteristic polynomial \( \lambda^3 + (4 - k_3)\lambda^2 + (3 - k_2)\lambda - k_1 \). To place the eigenvalues at 0, 1/2, -1/2, the desired characteristic polynomial is \( \lambda(\lambda - 1/2)(\lambda + 1/2) = \lambda^3 - 1/4 \la \). So we should choose \( k_1 = 0, k_2 = 13/4, k_3 = 4 \).