1 LC Tank: Diagonalization with complex eigenvalues

Consider the following circuit like you saw in lecture:

![Circuit Diagram]

This is sometimes called an LC tank and we will derive its response in this problem. Assume at $t = 0$ we have $V_C(0) = V_S = 1$ V and $\frac{dV_C}{dt}(t = 0) = 0$.

a) Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of $V_S$, $L$, and $C$.

b) Write the system of equations in vector/matrix form with the vector state variable $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. This should be in the form $\frac{d}{dt} \mathbf{x}(t) = A \mathbf{x}(t)$ with a $2 \times 2$ matrix $A$.

Find the initial conditions $\mathbf{x}(0)$. 
c) Find the eigenvalues of the $A$ matrix symbolically.

\[ \text{What are the eigenvectors associated with these eigenvalues?} \]

\[ \text{Use the eigenvalues and eigenvectors found above to diagonalize } A \text{ as } A = V \Lambda V^{-1} \text{ where } \Lambda \text{ is a diagonal matrix. Suppose } L = 9 \text{ nH and } C = 1 \text{ nF.} \]

\[ \text{Use a change of basis for the state variable } \tilde{x}(t) \text{ into } \tilde{z}(t) \text{ such that } \]
\[ \frac{d}{dt} \tilde{z}(t) = \Lambda \tilde{z}(t), \text{ and express the initial conditions } \tilde{z}(0) \]

\[ \text{Solve the differential equations in } \tilde{z}(t) \]
g) **Convert your solutions back to \( \tilde{x}(t) \). Plot \( V_C(t) \) and \( I_L(t) \).** What do you notice about the solutions? Are they complex functions? **HINT:** Remember 
\[ e^{j\theta} = \cos(\theta) + j\sin(\theta). \]

## 2 Complex Matrix Inverse

Consider a complex matrix

\[ M = M_r + jM_i \]

and its inverse

\[ N = N_r + jN_i \]

a) **Show that the inverse of \( \overline{M} = M_r - jM_i \) (the complex conjugate of \( M \)) is equal to \( \overline{N} = N_r - jN_i \) (the complex conjugate of \( N \)).**