1 Bicycle Kinematics

The kinematic equations are given by:

\[ \dot{x}_b = V \cos(\theta(t)) \]  
\[ \dot{y}_b = -V \sin(\theta(t)) \]  
\[ \dot{\theta} = \frac{V}{L} \tan(\delta(t)) \]  
\[ y_a = y_b - L \sin(\theta(t)) \]

For simplicity, we can assume that the vehicle speed \( V \) is constant. There is then just one control input the system, the steering angle \( \delta \), and we can consider the output to be \( y_a \), the road distance from the front axle. Now for following a straight track with a small heading error (say less than 20°), we can linearize the differential equations using \( \sin \theta \approx \theta \) and \( \cos \theta \approx 1 \). Thus we get:

\[ \dot{y}_b \approx -V \theta \]  
\[ \dot{\theta} \approx \frac{V}{L} \delta(t) \]  
\[ \dot{y}_a \approx \dot{y}_b - L \dot{\theta} = -V \theta - L \dot{\theta} \]

We would like to get a differential equation relating the input steering angle to the front axle position error. To do this, we differentiate eqn. 7 and substitute eqn. 6 for steering angle obtaining

\[ \ddot{y}_a = -\frac{V^2}{L} \delta(t) - V \dot{\delta}(t). \]

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_b )</td>
<td>X coordinate of midpoint of rear axle</td>
</tr>
<tr>
<td>( x_a )</td>
<td>X coordinate of midpoint of front axle</td>
</tr>
<tr>
<td>( y_b )</td>
<td>lateral displacement w.r.t. road centerline at rear axle</td>
</tr>
<tr>
<td>( y_a )</td>
<td>lateral displacement w.r.t. road centerline at front axle</td>
</tr>
<tr>
<td>( \delta )</td>
<td>steering angle</td>
</tr>
<tr>
<td>( L )</td>
<td>wheel base</td>
</tr>
<tr>
<td>( \theta )</td>
<td>relative yaw angle w.r.t. road centerline</td>
</tr>
<tr>
<td>( V )</td>
<td>vehicle speed</td>
</tr>
</tbody>
</table>

Table 1: Definition of Variables
2 Proportional Control

Let’s see what happens when we apply a steering control to the system proportional to position error:

$$\delta(t) = k_p y_a(t)$$  \hspace{1cm} (9)

Then the closed loop system has dynamics described by the second order linear differential equation:

$$\ddot{y}_a + V k_p \dot{y}_a(t) + \frac{V^2}{L} k_p y_a(t) = 0.$$  \hspace{1cm} (10)

Let’s re-write this second order differential equation in state variable form, letting $x_1 = y_a$ and $x_2 = \dot{y}_a$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{V^2}{L} k_p & -V k_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ \hspace{1cm} (11)

This is just a homogeneous equation of the form $\dot{x} = Ax$, so we know the solution is just:

$$x(t) = e^{At}x(0)$$ \hspace{1cm} (12)

where $e^{At}$ is a matrix exponential given by

$$e^{At} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ...$$  \hspace{1cm} (13)

The system will be stable if the real part of the eigenvalues of $A$ are less than 0. You can verify that the eigenvalues of $A$ are

$$\lambda_{1,2} = \frac{V}{2} \left(-k_p \pm \sqrt{k_p^2 - \frac{4k_p}{L}}\right)$$  \hspace{1cm} (14)