EE243 Advanced Electromagnetic Theory

Lec #4: Electrostatics (Green's Thm.),

- **Uniqueness**
- **Equivalent Sources**
- **Zero fields Outside Region**
- **Reciprocity**

Reading: Jackson 1.8-1.10, 2.7

Integral Representation

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'} \right] da'$$

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} + F(\bar{x},\bar{x}')$$

$$\nabla'^2 G(\bar{x},\bar{x}') = -4\pi\delta(\bar{x}-\bar{x}') \qquad \nabla'^2 F(\bar{x},\bar{x}') = 0$$

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|}$$

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')\frac{1}{|\bar{x}-\bar{x}'|}d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{|\bar{x}-\bar{x}'|}\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta}{\delta n'}\left(\frac{1}{|\bar{x}-\bar{x}'|}\right. \right] da'$$

Integral Representation: Boundary Conditions

Jackson 1.10

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'} \right] da'$$

- 2^{nd} order differential equations in general require two Boundary conditions
- Integral representation has both Φ and its normal derivative on the boundary
- But the integral representation also has both G and its normal derivative on the boundary
- Since Φ and G can represent different physical problems they can take on different boundary conditions

Dirichlet Boundary Conditions G = 0

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'} \right] da'$$

- G = 0 => Only surface term requiring Φ remains
- Example Potential on Sphere known and ρ(x) =0 in volume Jackson 2.6-2.7
- Green's Function for image charge in sphere

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} - \frac{a}{x'\left|\bar{x}-\frac{a^2}{x'^2}\bar{x}'\right|}$$

  - Evaluate normal derivative of G

$$\Phi(\bar{x}) = \frac{1}{4\pi}\int \Phi(a,\theta',\phi')\frac{a(x^2-a^2)}{\left(x^2+a^2-2ax\cos\gamma\right)^{3/2}}$$

Dirichlet Boundary Conditions G = 0

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'} \right] da'$$

- G = 0 => first surface term zero
- Example Φ = 0 **Grounded Sphere** and ρ(x) in volume
- Green's Function for image charge in sphere

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} - \frac{a}{x'\left|\bar{x} - \frac{a^2}{x'^2}\bar{x}'\right|}$$

- Note: a charge distribution exists on the sphere but no knowledge about it is needed

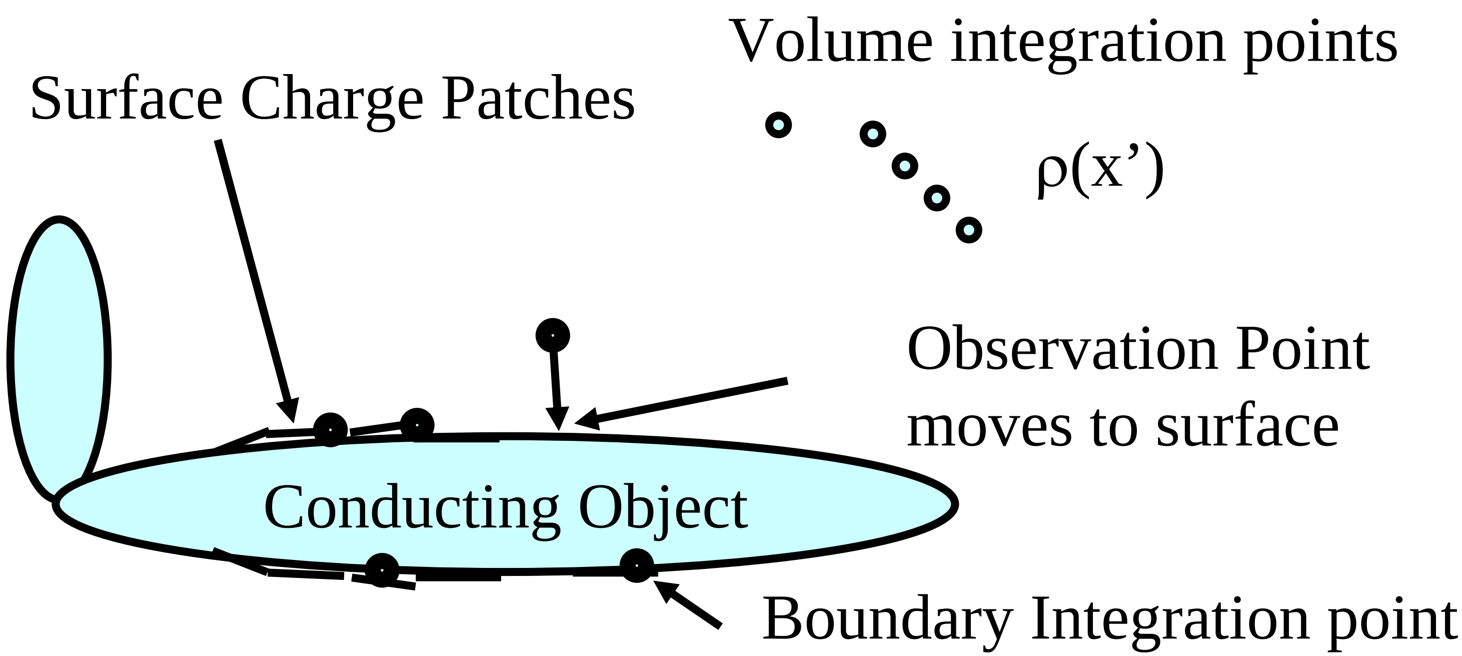
$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')\left[\frac{1}{|\bar{x}=\bar{x}'} - \frac{a}{x'\left|\bar{x} - \frac{a^2}{x'^2}\bar{x}'\right|} \right] d^3x'$$

Neuman Boundary Condition: dG/dn' = 0

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'} \right] da'$$

- Neuman dG/dn' = 0 => Only dΦ/dn' = $\sigma_{surface}$ remains
- It is hard to find examples of this Green's function
- If in addition $\sigma_{surface}$ = 0, all of the surface terms drop out
- Mixed Boundary Conditions
  - Location: Different BC can be used at different locations on surface and surface integral will still drop out
  - May also generalize for linear combination $\alpha\Phi + \beta\frac{\partial\Phi}{\partial n} = 0$

Integral Equation



N surface charge patches or unknowns
N observations points on surface where the potential is known
Each observation gives a different weighted sum of the charge patch contributions
N constraints on N unknowns => solve for N unknown surface patch charges

Integral Equation to Find Surface Charge

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S\left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'}\right]da'$$

- Example: Grounded Conducting Object and ρ(x)

  Φ = 0 => all of the F surface term drop out

  dΦ/dn'= $\sigma_{surface}$ remains

  Since Φ is known at every point on object restrict x to be on the object

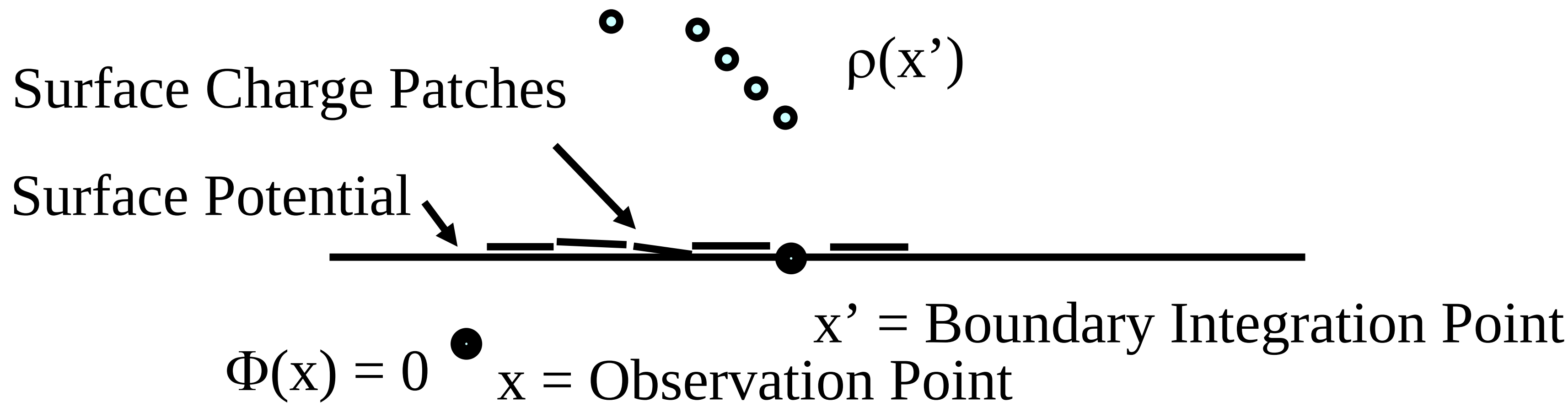
  Gives and integral equation for the surface charge

$$0 = \left[\frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S\left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'}\right]da'\right]_{\bar{x}\_on\_object}$$

  Generally the Green's function for free space is used

$$0 = \left[\frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')\frac{1}{|\bar{x}-\bar{x}'|}d^3x' + \frac{1}{4\pi}\oint_S\left[\frac{1}{|\bar{x}-\bar{x}'|}\frac{\delta\Phi}{\delta n'}\right]da'\right]_{\bar{x}\_on\_object}$$

Equivalence Theorem



When the observation point in the integral representation moves outside of the solution region the value of the potential is identically zero!

- This means that knowledge of the potential and its derivative normal to the boundary are sufficient **equivalent sources** to wall off the world outside the solution region!
- This implies that **it does not matter what is outside of the solution volume** and in fact it can be changed to simplify the solution and not affect the answer!
- This result is **true for any Green's function!**

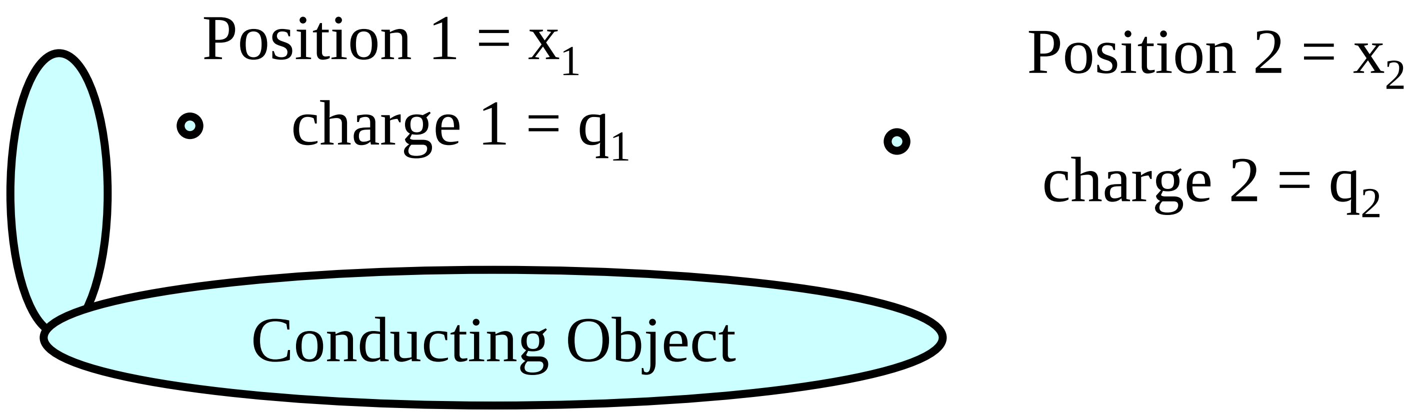
Uniqueness Theorem

The Electrostatic Solution is unique

Proof:

- Assume two, take difference and plug in Green's 1st Identity for both ϕ and ψ
- Difference is source free =>volume Laplacian integral = 0 (no sources)
- Difference is 0 on boundary => surface integral is zero
- Volume integral of gradient product is squared gradient and must integrate to zero => integrand must be zero everywhere in solution region.
- Hence the function (except for a constant off-set voltage) must be identical

Reciprocity



$\Phi(x_2,q_1) = \Phi(x_1,q_2)$

Proof:

- Green’s Theorem
- Poisson’s equation for $\Phi(x_2,q_1)$ and $\Phi(x_1,q_2)$ causes volume integral to give $\Phi(x_2,q_1) - \Phi(x_1,q_2)$
- In surface integral use homogeneous boundary condition to replace potential with derivative and integrand vanishes at every point on the boundary

Integral Equation Numerical Solution Method

$$0 = \left[\frac{1}{4\pi\varepsilon_0} \int_V \rho(\bar{x}') \frac{1}{|\bar{x} - \bar{x}'|} d^3x' + \frac{1}{4\pi} \oint_S \left[\frac{1}{|\bar{x} - \bar{x}'|} \frac{\delta\Phi}{\delta n'} \right] da' \right]_{\bar{x}\_on\_object}$$

- Expand surface charge as a function of position on surface as a sum of N linearly independent functions each with an unknown amplitude.
  - Example: finite basis functions or orthogonal functions
- For a given position on object evaluate the integral to get one constraint on the weighted sum of the unknown coefficients
- Repeat for N separate positions
- Solve the N constraints for the N unknowns coefficients using Matrix methods