EE243 Advanced Electromagnetic Theory

Lec # 7: Dielectric Materials

• Geometry for Homework Problem 3.1
• Multipoles
• Interaction of Multipoles
• Boundary Value Problems for Dielectrics
• Relationship of dielectric constant to molecules

Reading: Jackson Ch 4
Geometry HW 3.1

- Positive line source induces negative line sources on cylinders that can be approximated as being at their centers
- Find the potential at the observation point
- See if reciprocity holds when source and observation are interchanged
Overview

• Conductivity is produced by free carriers (holes and electrons) moving to surfaces.
• But charges (electrons) bound to nuclei (protons) can be displaced slightly by $E$.
• This creates dipole and possibly higher spherical expansion eigenfunction moments.
• They in turn create
  – Surface charge; reduction in internal field; internal lattice fields; stored energy; force on dielectrics
Dielectrics

• Charge produces E field
• E field produces Polarization and surface charge
• E field inside dielectric is lowered

charge 1 = q₁

Surface charges due to polarization P
Eigenfunctions in Cylindrical Coordinates

• x and y become ρ and φ
• Azimuthal = sin (mφ) and cos (mφ)
• Radial = Bessel functions J (oscillatory like)
• Hankel functions and Modified Bessel functions I and K (exp. like)
• Boundary Conditions
  – Wedge: Eigenfunction values of m (could be fractional)
  – Cylinders: Zeros of Bessel functions

Jackson pp. 112
Eigenfunctions in Spherical Coordinates

- x, y, z become \( \rho, \phi \) and \( \theta \)
- Azimuthal uniform = sinusoidal \( \phi \)
- Or azimuthal and \( \theta \) combine on sphere surface as spherical harmonics

\[
Y_{lm}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} P_l^m(\cos \theta) e^{im\phi}
\]

\[
g(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \phi)
\]

\[
A_{lm} = \int Y_{lm}^*(\theta, \phi) g(\theta, \phi) d\Omega
\]

\[
Y_{21}(\theta, \phi) = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}
\]  

Example

Jackson pp. 108
Spherical Harmonic Expansion

• Representation outside the charge distribution
• Take eigenfunction moments over distribution

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}
\]

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x')}{|x-x'|} \rho(x)
\]

\[
q_{lm} = \int Y_{lm}^*(\theta, \phi) r'^l \rho(x') d\Omega
\]

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{r} + \frac{p \cdot x}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \ldots \right]
\]

\[
Q_{i,j} = \int \left( 3x_i x_j - r'^2 \delta_{ij} \right) \rho(x') d^3 x'
\]
Results Spherical Harmonic Expansion

• Negative gradient gives $E_\rho$, $E_\theta$ and $E_\phi$

$$W = q\Phi(0) - p \cdot E(0) - \frac{1}{6} \sum_i \sum_j Q_{ij} \frac{\partial E_i}{\partial x_j}(0) + ...$$

• Energy

• Dipole case

$$E(x) = \frac{3n(p \cdot n) - p}{4\pi \varepsilon_0 |x - x_0|}$$

$$W_{12} = \frac{p_1 \cdot p_2 - 3(n \cdot p_1)(n \cdot p_2)}{4\pi \varepsilon_0 |x_1 - x_2|^3}$$

• For dipole the average over a spherical volume is the value at the center of the sphere
Ponderable Media

- Apply Maxwell’s equations locally and then average to get macroscopic ME.
- Curl $E = 0$ added up over volume remains the same => - Gradient of Potential
- Media free charges and dipole moments contribute to potential $P = \sum p_{\text{MOLECULE}}$
- Add up charge and dipole contributions
- Integrate dipole term by parts to move differential operator from $1/r$ to $P$.
- Obtain

Ponderable = having serious weight
Ponderable Media: Results

- **Div P** is like charge
- **Displacement**

\[
\nabla \cdot E = \frac{1}{\varepsilon_0} [\rho - \nabla \cdot P]
\]

\[
D = \varepsilon_0 E + P
\]

\[
\nabla \cdot D = \rho
\]

\[
\nabla \cdot E = \frac{\rho}{\varepsilon}
\]

- **Boundary Conditions**
  - **D** normal discontinuous
  - **E** tangential continuous

\[
(D_2 - D_1) \cdot n_{21} = \sigma
\]

\[
(E_2 - E_1) \times n_{21} = 0
\]
Image Charge Geometry

q’’ is necessary to create the fields for z < 0.
BVP Dielectrics: Image in Half-Space

• Charge \( q \) at \( z = a \) in material with \( \varepsilon_1 \) above half space of material with \( \varepsilon_2 \)
• Postulate two image charges
  – \( q' \) at \( z = -a \) for fields \( z > 0 \)
  – \( q'' \) at \( z = a \) for fields \( z < 0 \)
• Match \( D \) normal and \( E \) tangential
• Find

\[
q' = -\left( \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \right) q
\]

\[
q'' = \left( \frac{2\varepsilon_1}{\varepsilon_2 + \varepsilon_1} \right) q
\]

\[
\sigma_{POL} = -(P_2 - P_1) \cdot n_{21} = \frac{q}{2\pi} \frac{\varepsilon_0}{\varepsilon_1} \left( \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \right) \frac{a}{(a^2 + \rho^2)^{3/2}}
\]
Integral Representations & Equations

**Surface Charge Patches**

\[ \Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_1} \int_V \rho(x') G(\bar{x}, \bar{x}') d^3 x' + \frac{1}{4\pi} \int_S \left[ G(\bar{x}, \bar{x}') \frac{\delta \Phi}{\delta n'} - \Phi(\bar{x}') \frac{\delta G(\bar{x}, \bar{x}')}{\delta n'} \right] d\alpha' \]

**Volume integration points**

\[ \rho(\bar{x}') \]

**Observation Point outside moves to surface**

**Observation Point inside moves to surface**

**Boundary Integration point**

\[ \Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_2} \int_V \rho(x') G(\bar{x}, \bar{x}') d^3 x' + \frac{1}{4\pi} \int_S \left[ G(\bar{x}, \bar{x}') \frac{\delta \Phi}{\delta n'} - \Phi(\bar{x}') \frac{\delta G(\bar{x}, \bar{x}')}{\delta n'} \right] d\alpha' \]

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Integral Representations for Dielectrics

• Separate representations inside and outside homogeneous dielectrics
  – Use a free space Green’s function for the media in which the observation point is immersed
  – Will require knowledge of both the potential and its normal derivative on the boundary
Integral Equations for Dielectrics

- Use Integral Representations
- Form integral equations by
  - Requiring potential to be continuous and $D$ normal discontinuous by $\text{div} \ P$.
- Solve integral equations
  - Every point has two unknowns (potential and $\text{Div} \ P$) and two boundary conditions (potential continuous and $D$ normal continuous)
BVP Dielectrics: Sphere

- Spherical Harmonic Expansion inside and outside sphere with $m = 0$.
- Match Tangential $E$ and normal $D$
- Result
  - Constant and lower $E$ field in sphere
  - Only moment is dipole = $p$ times volume
  - $\text{Div } P$ gives surface charge
    \[ E_{IN} = \frac{3}{\varepsilon / \varepsilon_0 + 2} E_{APPLIED} \]
- Hole in Dielectric
  - Similar result
    \[ P = 3\varepsilon_0 \left( \frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right) E_{APPLIED} \]
    \[ \sigma_{POL} = 3\varepsilon_0 \left( \frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right) E_{APPLIED} \cos \theta \]
Molecular Level Models

• Electric field strength inside a material must be corrected
  – Macroscopic $P$ (lowers)
  – Near molecules add a discrete sum (usually zero)

• Electric Susceptibility

• Molecular Polarizability

\[
P = N\gamma_{\text{mol}} \left( \varepsilon_0 E + \frac{1}{3} P \right)
\]

\[
P = \varepsilon_0 \chi_e E
\]

\[
\chi_e = \frac{N\gamma_{\text{mol}}}{1 - \frac{1}{3} N\gamma_{\text{mol}}}
\]

\[
\gamma_{\text{mol}} = \frac{3}{N} \left( \frac{\varepsilon}{\varepsilon_0} - 1 \right)
\]

\[
\gamma_{\text{mol}} = \frac{3}{N} \left( \frac{\varepsilon}{\varepsilon_0} + 2 \right)
\]
Models for Molecular Polarizability

• Harmonic oscillator model for atom
• Displacement times charge is dipole
• Observations
  – Induced dipole moment is temperature independent
  – Partial orientation of otherwise random permanent dipole moments has higher molecular polarizability that reduces with temperature increase
Energy and Force

- Start charge based
- Substitute for charge
- Integrate by parts
- Chain rule derivative

\[ \delta W = \int \delta \rho(x) \Phi(x) d^3 x \]
\[ \delta \rho = \nabla \cdot (\delta D) \]
\[ \delta W = \int E \cdot \delta D d^3 x \]
\[ E \delta D = \frac{1}{2} \delta (E \cdot D) \]
\[ W = \frac{1}{2} \int E \cdot D d^3 \]
Energy with Motion of Dielectric

\[ W = \frac{1}{2} \int E \cdot D d^3 x \]

\[ w = -\frac{1}{2} P \cdot E \]

\[ \delta W = \frac{1}{2} \int (\rho \delta \Phi + \Phi \delta \rho) d^3 x \]

\[ \delta W_1 = \frac{1}{2} \int (\rho \delta \Phi_1) d^3 x \]

\[ \delta W_2 = \frac{1}{2} \int (\rho \delta \Phi_2 + \Phi \delta \rho_2) d^3 x = -2 \delta W_1 \]

Dielectrics are attracted to high field regions and fields are attracted to dielectric regions

- Basic formula
- Result: Insert a dielectric into a field with fixed sources
- Extend for fixed Voltages as two steps
  - Without batteries
  - Reconnect batteries