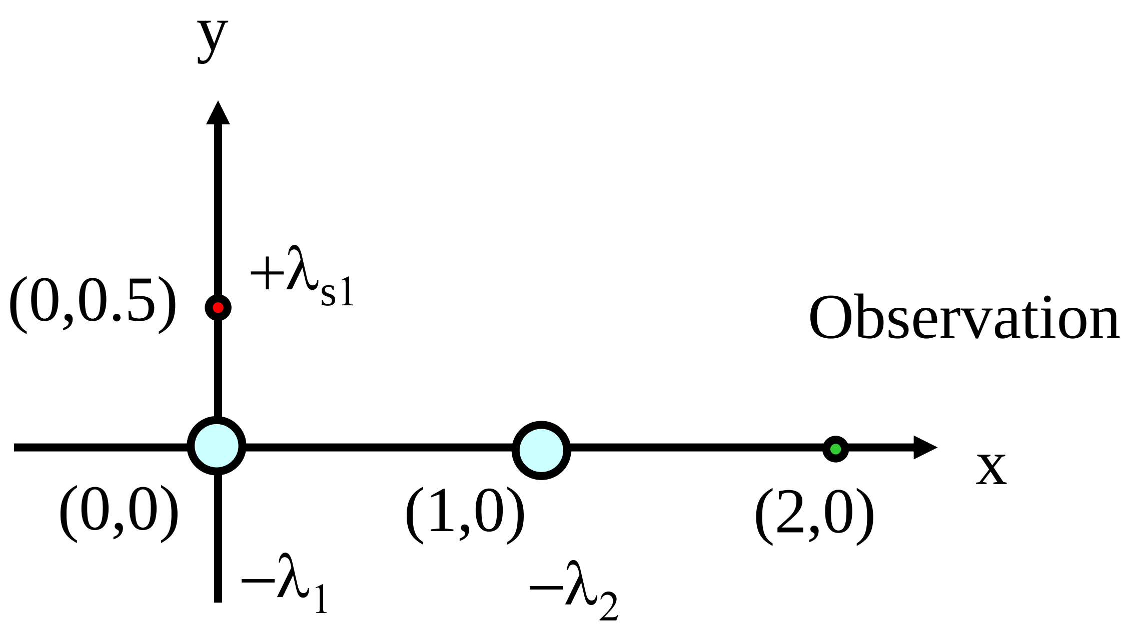
EE243 Advanced Electromagnetic Theory

Lec # 7: Dielectric Materials

- Geometry for Homework Problem 3.1
- Multipoles
- Interaction of Multipoles
- Boundary Value Problems for Dielectrics
- Relationship of dielectric constant to molecules

Reading: Jackson Ch 4

Geometry HW 3.1

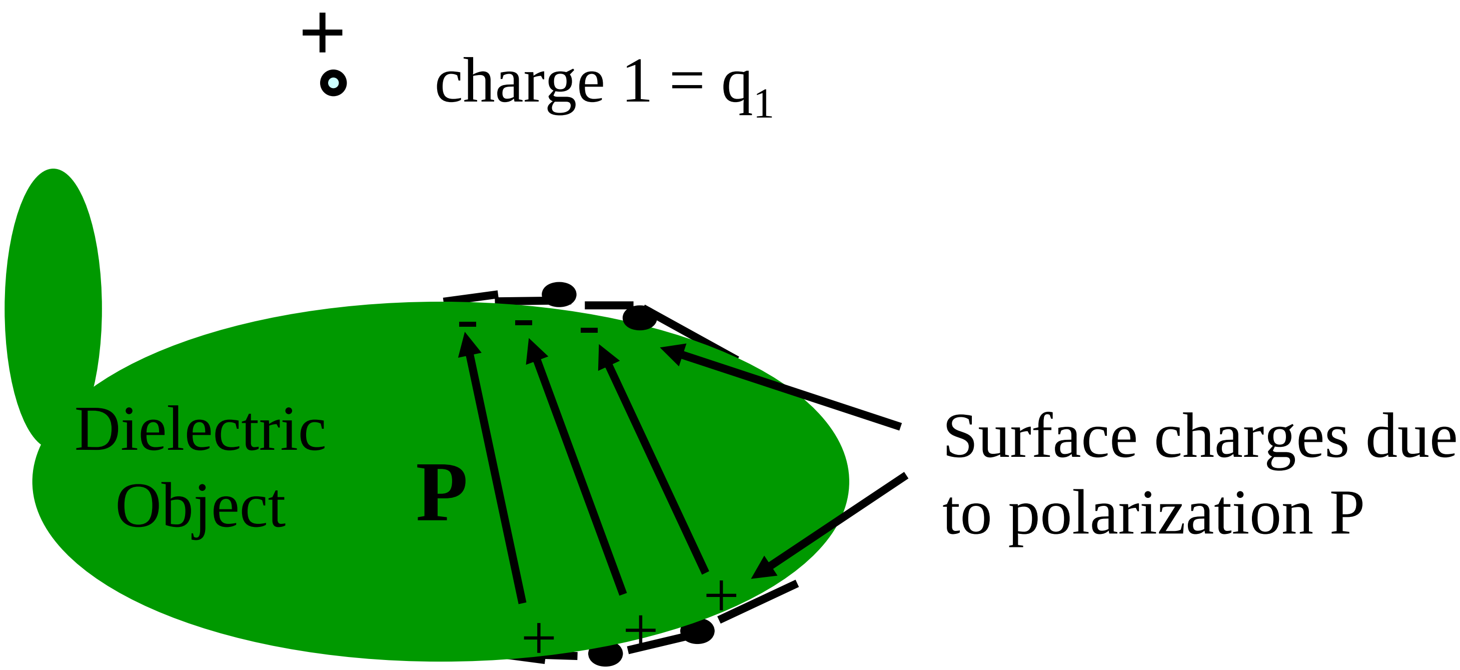


- Positive line source induces negative line sources on cylinders that can be approximated as being at their centers
- Find the potential at the observation point
- See if reciprocity holds when source and observation are interchanged

Overview

- Conductivity is produced by free carriers (holes and electrons) moving to surfaces.
- But charges (electrons) bound to nuclei (protons) can be displaced slightly by E.
- This creates dipole and possibly higher spherical expansion eigenfunction moments.
- They in turn create
  - Surface charge; reduction in internal field; internal lattice fields; stored energy; force on dielectrics

Dielectrics



- Charge produces E field
- E field produces Polarization and surface charge
- E field inside dielectric is lowered

Eigenfunctions in Cylindrical Coordinates

Jackson pp. 112

- x and y become ρ and ϕ
- Azimuthal = sin ($m\phi$) and cos ($m\phi$)
- Radial = Bessel functions J (oscillatory like)
- Hankel functions and Modified Bessel functions I and K (exp. like)
- Boundary Conditions
  - Wedge: Eigenfunction values of m (could be fractional)
  - Cylinders: Zeros of Bessel functions

Eigenfunctions in Spherical Coordinates

Jackson pp. 108

- x, y, z become ρ, ϕ and θ
- Azimuthal uniform = sinusoidal ϕ
- Or azumuthal and θ combine on sphere surface as spherical harmonics

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{i} A_{lm} Y_{lm}(\theta,\phi)$$

$$A_{lm} = \int Y_{lm}^{*}(\theta,\phi) g(\theta,\phi) d\Omega$$

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi}$$

Example

Spherical Harmonic Expansion

- Representation outside the charge distribution
- Take eigenfunction moments over distribution

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{i} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x^{'})}{|x - x^{'}|} \rho(x)$$

$$q_{lm} = \int Y_{lm}^{\ *}(\theta,\phi) r^{'l} \rho(x^{'}) d\Omega$$

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q}{r} + \frac{p \cdot x}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \ldots \right]$$

$$Q_{i,j} = \int \left(3x_i^{'} x_j^{'} - r^{'2} \delta_{ij} \right) \rho(x^{'}) d^3 x^{'}$$

Results Spherical Harmonic Expansion

- Negative gradient gives E_ρ, E_θ and E_ϕ
- Energy $$W = q\Phi(0) - p \cdot E(0) - \frac{1}{6}\sum_i \sum_j Q_{ij} \frac{\partial E_i}{\partial x_j}(0) + \dots$$
- Dipole case $$E(x) = \frac{3n(p \cdot n) - p}{4\pi\varepsilon_0 \,|\, x - x_0 \,|}$$

$$W_{12} = \frac{p_1 \cdot p_2 - 3(n \cdot p_1)(n \cdot p_2)}{4\pi\varepsilon_0 \,|\, x_1 - x_2 \,|^3}$$

- For dipole the average over a spherical volume is the value at the center of the sphere

Ponderable Media

- Apply Maxwell's equations locally and then average to get macroscopic ME.
- Curl E = 0 added up over volume remains the same => - Gradient of Potenetial
- Media free charges and dipole moments contribute to potential P = sum $p_{MOLECULE}$
- Add up charge and dipole contributions
- Integrate dipole term by parts to move differential operator from 1/r to P.
- Obtain

Ponderable = having serious weight

Ponderable Media: Results

- Div P is like charge
- Displacement

$$\nabla \cdot E = \frac{1}{\varepsilon_0}[\rho - \nabla \cdot P]$$

$$D = \varepsilon_0 E + P$$

$$\nabla \cdot D = \rho$$

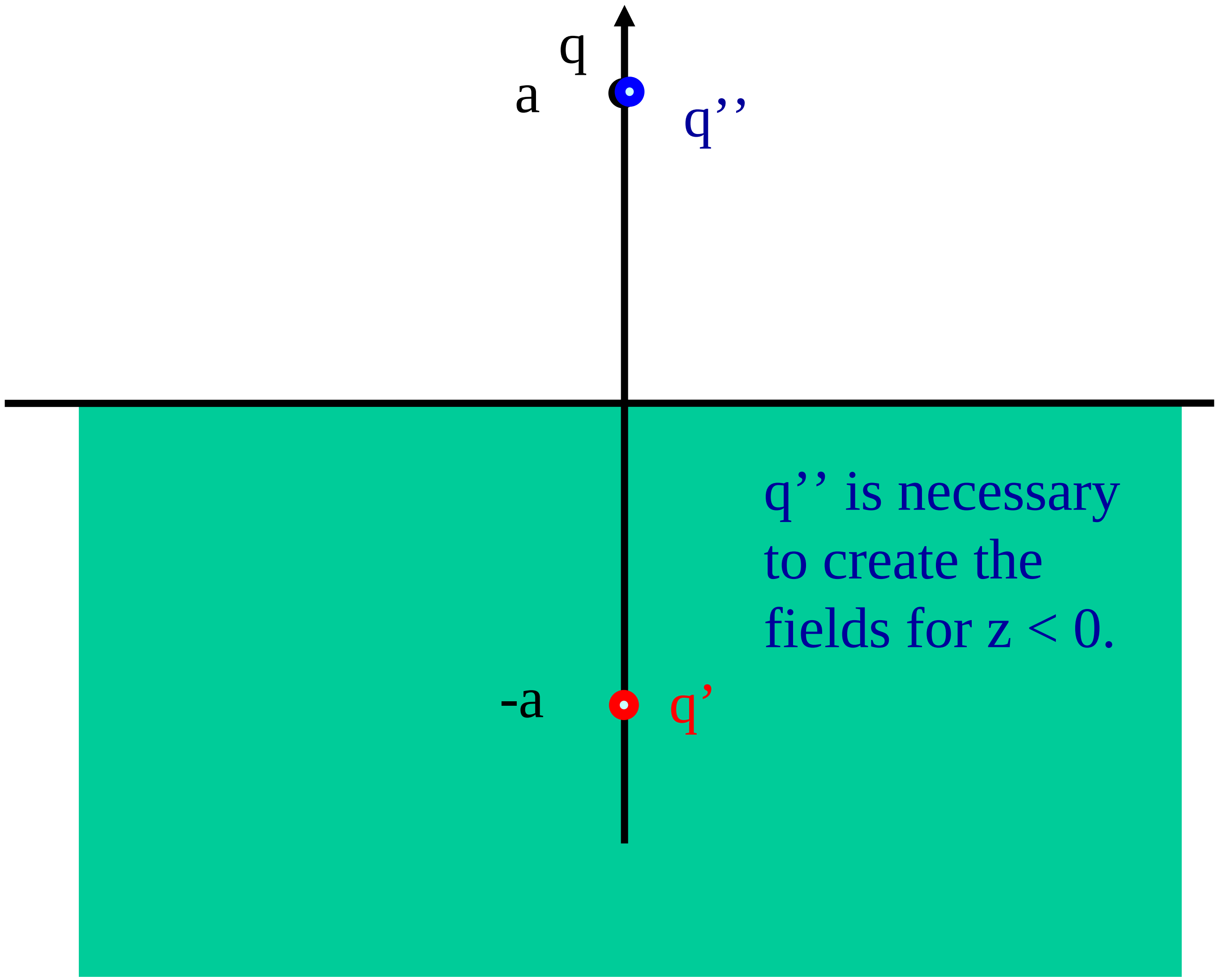
$$\nabla \cdot E = \frac{\rho}{\varepsilon}$$

- Boundary Conditions
  - D normal discontinuous
  - E tangential continuous

$$(D_2 - D_1) \cdot n_{21} = \sigma$$

$$(E_2 - E_1) \times n_{21} = 0$$

Image Charge Geometry



q'' is necessary to create the fields for z < 0.

BVP Dielectrics: Image in Half-Space

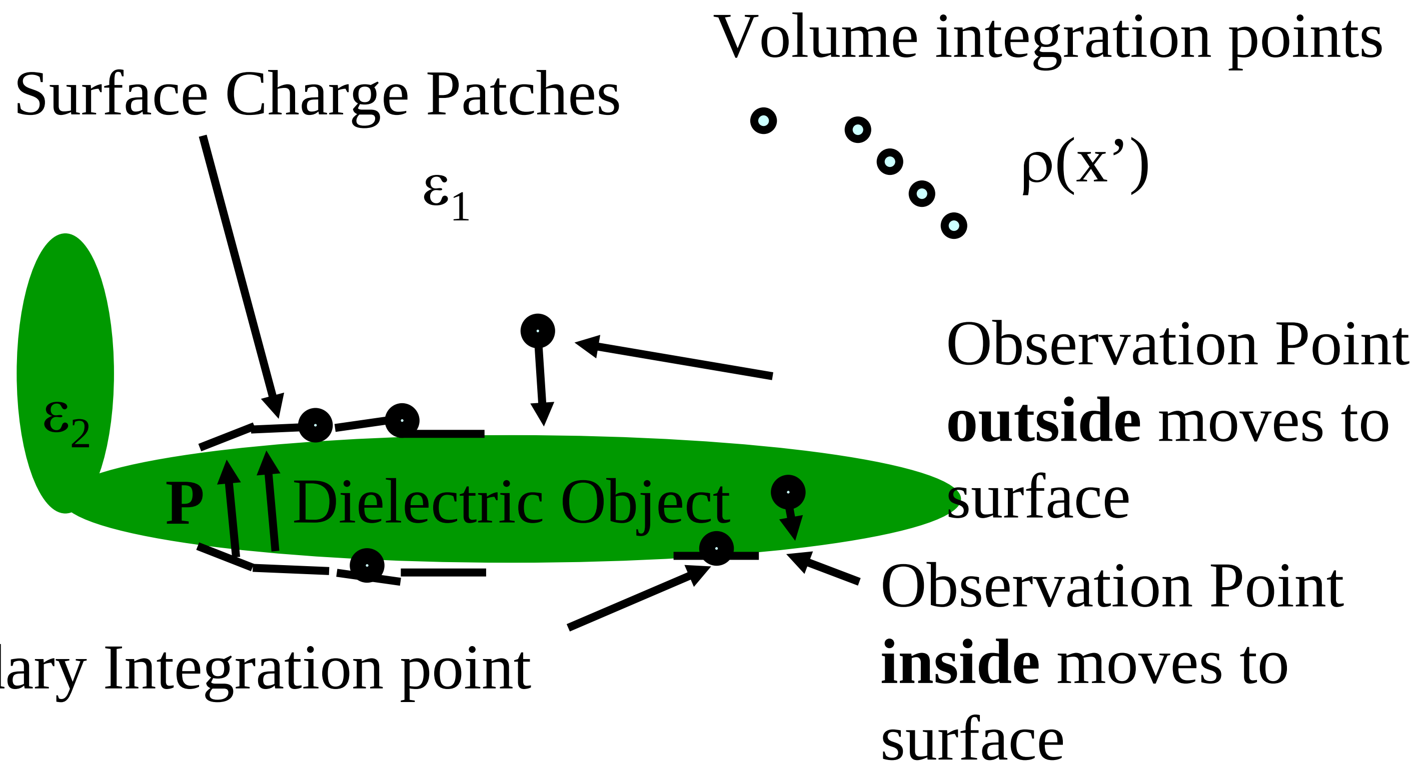
- Charge q at z = a in material with ε_1 above half space of material with ε_2
- Postulate two image charges
  - q' at z = -a for fields z > 0
  - q'' at z = a for fields z < 0
- Match D normal and E tangential
- Find

$$q' = -\left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}\right) q$$

$$q'' = \left(\frac{2\varepsilon_1}{\varepsilon_2 + \varepsilon_1}\right) q$$

$$\sigma_{POL} = -(P_2 - P_1) \cdot n_{21} = \frac{q}{2\pi} \frac{\varepsilon_0}{\varepsilon 1} \left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}\right) \frac{a}{\left(a^2 + \rho^2\right)^{3/2}}$$

Integral Representations & Equations



$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_1}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'} \right] da'$$

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_2}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'} \right] da'$$

Integral Representations for Dielectrics

- Separate representations inside and outside homogeneous dielectrics
  - Use a free space Green's function for the media in which the observation point is immersed
  - Will require knowledge of both the potential and its normal derivative on the boundary

Integral Equations for Dielectrics

- Use Integral Representations
- Form integral equations by
  - Requiring potential to be continuous and D normal discontinuous by div P.
- Solve integral equations
  - Every point has two unknowns (potential and Div P) and two boundary conditions (potential continuous and D normal continuous)

BVP Dielectrics: Sphere

- Spherical Harmonic Expansion inside and outside sphere with m = 0.
- Match Tangential E and normal D
- Result
  - Constant and lower E field in sphere
  - Only moment is dipole = p times volume
  - Div P gives surface charge
- Hole in Dielectric
  - Similar result

Jackson 158

$$E_{IN} = \frac{3}{\varepsilon / \varepsilon_0 + 2} E_{APPLIED}$$

$$P = 3\varepsilon_0 \left(\frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right) E_{APPLIED}$$

$$\sigma_{POL} = 3\varepsilon_0 \left(\frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right) E_{APPLIED} \cos\theta$$

Molecular Level Models

- Electric field strength inside a material must be corrected
  - Macroscopic P (lowers)
  - Near molecules add a discrete sum (usually zero)
- Electric Susceptibility
- Molecular Polarizability

$$P = N\gamma_{mol}\left(\varepsilon_0 E + \frac{1}{3}P\right)$$

$$P = \varepsilon_0 \chi_e E$$

$$\chi_e = \frac{N\gamma_{mol}}{1 - \frac{1}{3}N\gamma_{mol}}$$

$$\gamma_{mol} = \frac{3}{N}\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)$$

Models for Molecular Polarizability

- Harmonic oscillator model for atom
- Displacement times charge is dipole
- Observations
  - Induced dipole moment is temperature independent
  - Partial orientation of otherwise random permanent dipole moments has higher molecular polarizability that reduces with temperature increase

Energy and Force

- Start charge based
- Substitute for charge
- Integrate by parts
- Chain rule derivative

$$\delta W = \int \delta\rho(x)\Phi(x)d^3x$$

$$\delta\rho = \nabla\cdot(\delta D)$$

$$\delta W = \int E\cdot\delta D d^3x$$

$$E\delta D = \frac{1}{2}\delta(E\cdot D)$$

$$W = \frac{1}{2}\int E\cdot D d^3$$

Energy with Motion of Dielectric

$$W = \frac{1}{2}\int E \cdot D d^3x$$

$$w = -\frac{1}{2} P \cdot E$$

$$\delta W = \frac{1}{2}\int \left(\rho\delta\Phi + \Phi\,\delta\rho\right) d^3x$$

$$\delta W_1 = \frac{1}{2}\int \left(\rho\delta\Phi_1\right) d^3x$$

$$\delta W_2 = \frac{1}{2}\int \left(\rho\delta\Phi_2 + \Phi\,\delta\rho_2\right) d^3x = -2\delta W_1$$

- Basic formula
- Result: Insert a dielectric into a field with fixed sources
- Extend for fixed Voltages as two steps
  - Without batteries
  - Reconnect batteries

Dielectrics are attracted to high field regions and fields are attracted to dielectric regions