EE243 Advanced Electromagnetic Theory

Lec # 12: Waves in Dispersive Media

• Models for Medial (harmonic oscillator)
• Behavior of permittivity and refractive index vs \( \omega \)
• Superposition of waves and group velocity
• Pulse width increase with propagation
• Implications of Causality and initial/final condition
• Kramers Kronig Relations
• \( \omega - \beta \) diagrams

Reading: Jackson Ch 7.8-7.10 (skip 7.6 and 7.7)
Overview

• A harmonic oscillator model of electrons circulating about the nucleus is the basic phenomena by which materials affect EM waves.

• These ponderable media and also boundary (eigenvalue) constraints produce wave phase velocities that depend on frequency.

• This so called dispersion generally makes
  – the energy velocity less than the speed of light and (even undefined)
  – And causes the pulse length to increase with distance.

• $\varepsilon(\omega)/\varepsilon_0$ is an analytical function and the real part can be found from the imaginary part and visa versa

• The tool for characterizing wave dispersion is the $\omega$-$\beta$ diagram that plots the radian frequency versus the propagation $k$-vector.
Harmonic Oscillator Model for Material

\[ m[\ddot{x} + \gamma \dot{x} + \omega_0^2] = -e\overline{E}(\bar{x}, t) \]

\[ \bar{p} = -e\bar{x} = \frac{e^2}{m} \frac{E}{\omega_0^2 - i\omega\gamma - \omega^2} \]

\[ \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_j f_j \frac{1}{\omega_i^2 - i\omega\gamma_i - \omega^2} \]

- 2nd order differential equation in time
- \( e^{i\omega t} \) converts to 2nd order algebraic equation
- Polarization is charge times displacement
- Dielectric constant is 1 + polarization effects
- Add contributions of each oscillator type
Permittivity Frequency Behavior

- $\varepsilon(\omega)$ has frequency dependence
- $\text{Re } \varepsilon(\omega)$ generally decreases with increasing frequency
- $\text{Im } \varepsilon(\omega)$ shows resonate peaks
- With low damping see resonate absorption and anomolous dispersion [negative slope of $\varepsilon(\omega)$]
- Represent $\varepsilon(\omega)$ by sum of poles in complex $\omega$ plane
- Free electrons give pole at zero (Drude Model)
- At high frequency converges to 1 as constant/$\omega^2$
Refractive Index Water Versus Frequency

- Real part drops from 9 to 1.5 and then 1.0
- Imaginary part rises to peak, drops to valley (visible), rises to peak, and drops to valley.
- First peak is due to vibrational modes in molecules and interaction among molecules.
- Second valley is due to electronic states of outer and then core electrons.

Plasmons on metals come into this picture when $\varepsilon(\omega)$ is negative near the real axis (free electrons responding to $E$).

$\lambda = \frac{1240nm}{hv}$

13.5$nm \rightarrow 92eV$
193$nm \rightarrow 6.4eV$
590$nm \rightarrow 2.1eV$
1240$nm \rightarrow 1eV$
Dispersion Single Wave Propagation

\[- \vec{k} \cdot \vec{k} + \omega^2 \mu(\omega) \varepsilon(\omega) = 0\]

\[|k| = k_0 = \omega \sqrt{\mu(\omega) \varepsilon(\omega)}\]

\[k_0 = \sqrt{\varepsilon_r(\omega)} \omega \mu_0 \varepsilon_0\]

\[k_0 = n(\omega) \frac{\omega}{c}\]

\[(k_0 x - \omega t) = 0\]

\[v_p = \frac{x}{t} = \frac{\omega}{k} = \frac{\omega}{n(\omega) \frac{\omega}{c}} = \frac{c}{n(\omega)}\]

- Wave equation constraint on propagation vector
- \((kx-\omega t)=0 \Rightarrow v_p=w/k\)
- Length of k-vector determines phase velocity
- Not all signal components remain in phase
Group Velocity Derivation

• Useful information or energy has more than one frequency to have a finite duration

• Consider Fourier Transform representation and use $\omega(k) = \omega(-k)$

• Choose a finite envelope and find frequency spread $\Delta x \Delta k > 1/2$

• Assume $\omega(k)$ approximated as linear
  – Constant $\Rightarrow$ phase shift
  – $\omega \frac{d\omega}{dk}$ gives delay $\Rightarrow v_g = \frac{d\omega}{dk}$

• Curvature gives pulse broadening

$$v_g = \frac{c}{n(\omega) + \omega \left( \frac{dn}{d\omega} \right)}$$
Causality

\[ D(\bar{x}, t) = \varepsilon_0 \left\{ E(\bar{x}, t) + \int_{-\infty}^{+\infty} G(\tau) E(\bar{x}, t-\tau) d\tau \right\} \]

\[ G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ \varepsilon(\omega)/\varepsilon_0 - 1 \right] e^{-i\omega \tau} d\omega \]

\[ \varepsilon(\omega)/\varepsilon_0 = 1 + \int_{0}^{\infty} G(\tau) e^{i\omega \tau} d\tau \]

- D is the response to E and \( D(\omega) = e(\omega)E(\omega) \)
- Write as summation over time
- Kernal is the polarizability and has duration of \( \gamma^{-1} \).
- Must be careful when time is large and mean free path involves multiple neighbors (anomalous skin effects)
- \( \varepsilon(\omega)/\varepsilon_0 \) is an analytic function in the upper half-plane
Kramers-Kronig Relations

\[ \text{Re} \varepsilon(\omega)/\varepsilon_0 = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega \text{Im} \varepsilon(\omega')/\varepsilon_0}{\omega'^2 - \omega^2} d\omega' \]

\[ \text{Im} \varepsilon(\omega)/\varepsilon_0 = -\frac{2\omega}{\pi} P \int_0^\infty \frac{[\text{Re} \varepsilon(\omega')/\varepsilon_0 - 1]}{\omega'^2 - \omega^2} d\omega' \]

- Analytic => Cauchy contour integral
- Part at infinity gives zero
- Integrate out singularity at pole
- P is principle value at pole
Boundary Conditions Create Dispersion

\[-\bar{k} \cdot \bar{k} + \omega^2 \mu \varepsilon = 0\]

\[-k_z k_z - \frac{(2\pi)^2}{a^2} - \frac{(2\pi)^2}{b^2} + \omega^2 \mu \varepsilon = 0\]

\[-k_z k_z + \omega^2 \mu \varepsilon [1 - \frac{a^2}{\omega^2 \mu \varepsilon} \frac{b^2}{\omega^2 \mu \varepsilon}] = 0\]

\[\varepsilon_r(\omega) = [1 - \frac{a^2}{\omega/c)^2} \frac{b^2}{(\omega/c)^2}]\]

• Boundary conditions contribute eigenvalues
• Wave equation forces constraint that gives the dispersion relationship
• Could interpret as relative permittivity
$\omega - \beta$ or $k - \beta$ Diagram (1-D)

- $\omega$ normalized to $k$-vector units

- $k_0 = \omega / c$

- $k$-vector in propagation direction

- $\beta = k$

- plot $\omega$ versus $k$ dispersion relationship

- Speed of light reference $k_0 = \omega / c$

- Phase velocity = global slope

- Group velocity = local slope
Midterm Exam

• In Class Tuesday October 24th
• Covers material through today (Chapter 7)
• Open Book, Open Notes, Bring Calculator, Paper Provided

• Topics
  – Green’s functions free space and use in Theorems and concepts with emphasis on statics
  – Separation of variables in rectangular coordinates using N-1 and N method
  – Time-Harmonic ME, planewaves, boundary conditions, and dispersion