EE243 Advanced Electromagnetic Theory

Lec # 13: Waveguides and sources

- **Source Free Region: Vector Potentials A and F**
- **Single direction component of A and F**
  - **Give TM and TE**
  - **Are Adequate**
- **Perfectly Electrically Conducting Guides**
- **Source Excitation via Reciprocity**

Reading: Jackson Ch 8.2-8.4, 8.12
Harrington 3.2, 3.12
Collin 4.10

Overview

- A very general TE/TM separation into two scalar problems is first developed as a working tool.
- This TE/TM separation with respect to the z (propagation) direction is applied to perfectly electrically conducting waveguides.
- The waveguide modes, phase and group velocities are then derived.
- General representations for fields in waveguides are then developed.
- Reciprocity is then used to determine the excitation of a given waveguide mode.

Overview: TE/TM Separation

- In a homogeneous source free region, it is adequate in general to independently solve for TWO SCALAR FUNCTIONS instead of 6 components simultaneously (3 components of E and 3 components of B).
- These scalar functions are the components of the vector potentials A and F in a common direction.
- Under the Lorenz Gauge these scalar functions satisfy the wave equation and their boundary conditions generally differ.
- The scalar component of A gives fields that are transverse magnetic and the scalar component of F gives the fields that are transverse magnetic (both with respect to the common direction).

Magnetic/Electric Duality

Harrington 3.2

Electric Sources

$$\nabla \times \overline{H} = -i\omega\varepsilon\overline{E} + \overline{J}$$

$$\nabla \times \overline{E} = i\omega\mu\overline{H}$$

$$\overline{H} = \nabla \times \overline{A}$$

$$\overline{\mathrm{A}} = \frac{1}{4\pi}\int_V \frac{\overline{J}e^{ik\left|\overline{x}-\overline{x}'\right|}}{\left|\overline{x}-\overline{x}'\right|} d^3x'$$

Magnetic Sources

$$\nabla \times \overline{E} = i\omega\mu\overline{H} - \overline{M}$$

$$\nabla \times \overline{H} = -i\omega\varepsilon\overline{E}$$

$$\overline{E} = -\nabla \times \overline{F}$$

$$\overline{\mathrm{F}} = \frac{1}{4\pi}\int_V \frac{\overline{M}e^{ik\left|\overline{x}-\overline{x}'\right|}}{\left|\overline{x}-\overline{x}'\right|} d^3x'$$

- Dual equations for problems in which only an electric source J or only a magnetic source M are present.

Simultaneous Electric and Magnetic Sources

$$\overline{E} = -\nabla \times \overline{F} + \frac{1}{i\omega\varepsilon}\left(\nabla \times \nabla \times \overline{A} - \overline{J}\right)$$

$$\overline{H} = \nabla \times \overline{A} + \frac{1}{i\omega\mu}\left(\nabla \times \nabla \times \overline{F} - \overline{M}\right)$$

$$\overline{\mathrm{A}} = \frac{1}{4\pi}\int_V \frac{\overline{J} e^{ik\left|\overline{x}-\overline{x}'\right|}}{\left|\overline{x}-\overline{x}'\right|} d^3x'$$

$$\overline{\mathrm{F}} = \frac{1}{4\pi}\int_V \frac{\overline{M} e^{ik\left|\overline{x}-\overline{x}'\right|}}{\left|\overline{x}-\overline{x}'\right|} d^3x'$$

- Superimpose contributions for the two source types.

Source Free Region

Harrington 3.12

$$\overline{E} = -\nabla \times \overline{F} + \frac{1}{i\omega\varepsilon}\left(\nabla \times \nabla \times \overline{A}\right)$$

$$\overline{H} = \nabla \times \overline{A} + \frac{1}{i\omega\mu}\left(\nabla \times \nabla \times \overline{F}\right)$$

- Representation is still valid in a source free regions (A and F are viewed as being produced by sources outside of the region).

- With the Lorenz gauge A and F satisfy the wave equation and the fields are give by

$$\nabla^2 \overline{A} + k^2 \overline{A} = 0$$

$$\nabla^2 \overline{F} + k^2 \overline{F} = 0$$

$$\overline{E} = -\nabla \times \overline{F} + i\omega\mu\overline{A} + \frac{1}{i\omega\varepsilon}\nabla\left(\nabla \cdot \overline{A}\right)$$

$$\overline{H} = \nabla \times \overline{A} + i\omega\overline{F} + \frac{1}{i\omega\mu}\nabla\left(\nabla \cdot \overline{F}\right)$$

Only Electric Potential in z Direction

$$\overline{A} = \psi\hat{z}$$

$$E_x = \frac{1}{-i\omega\varepsilon}\frac{\partial^2\psi}{\partial x\partial z}$$

$$E_y = \frac{1}{-i\omega\varepsilon}\frac{\partial^2\psi}{\partial y\partial z}$$

$$E_z = \frac{1}{-i\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

$$H_x = \frac{\partial\psi}{\partial y}$$

$$H_y = -\frac{\partial\psi}{\partial x}$$

$$H_z = 0$$

- This contribution is transverse magnetic (TM) to the z direction

Only Magnetic Potential in z Direction

$$\overline{F} = \psi \hat{z}$$

$$E_x = -\frac{\partial \psi}{\partial y}$$

$$E_y = \frac{\partial \psi}{\partial x}$$

$$E_z = 0$$

$$H_x = \frac{1}{-i\omega\mu} \frac{\partial^2 \psi}{\partial x \partial z}$$

$$H_y = \frac{1}{-i\omega\mu} \frac{\partial^2 \psi}{\partial y \partial z}$$

$$H_z = \frac{1}{-i\omega\mu} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \psi$$

- This contribution is transverse electric (TE) to the z direction

Total Solution

- The addition of the TM and TE solutions is sufficient to express any arbitrary field in a source free region.
- Further, the choice of the direction about which to make the TE/TM can be any direction.
  - In empty waveguides and on dielectric slabs the TE/TM separation is often the direction of propagation.
  - An exception is a partially filled waveguide where the separation directions is perpendicular to the dielectric filling.
- The scalar function describing the TE and TM solutions typically have different boundary conditions
  - Perfect electric conductor

$$E_z \,|_S = 0$$

$$\frac{\partial B_z}{\partial n} \,|_S = 0$$

ME for Uniform Guided Waves

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + ik_z\phi\hat{z}$$

Jackson 8.2

$$\nabla\cdot\overline{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + ik_zV_z$$

$$\nabla\times\overline{V} = \left(\frac{\partial V_z}{\partial y} - ik_zV_y\right)\hat{x} + \left(ik_zV_x - \frac{\partial V_z}{\partial x}\right)\hat{y} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - k^2\phi = (\nabla_t^2 - k^2)\phi$$

- Guided waves have a uniform longitudinal phase variation e^{ikz}
- Factor the differential operators into transverse operators and longitudinal algebraic multipliers.

ME for Uniform Guided Waves (Cont.)

$$\left[\nabla_t^2 + \left(\mu\varepsilon\omega^2 - k^2\right)\right]\overline{E} = 0$$

Jackson 8.2

$$\overline{E} = E_z\hat{z} + \overline{E}_t$$

$$\overline{E}_t = \frac{1}{\left(\mu\varepsilon\omega^2 - k^2\right)}\left[k\nabla_t E_z - \omega\hat{z}\times\nabla_t B_z\right]$$

$$\overline{B}_t = \frac{1}{\left(\mu\varepsilon\omega^2 - k^2\right)}\left[k\nabla_t B_z + \mu\varepsilon\omega\hat{z}\times\nabla_t E_z\right]$$

- E and B satisfy wave equation with transverse operator and $-k^2$
- Break up E and B into longitudinal and transverse
- Transverse fields can be found from E_z and B_z.

TEM Degeneracy

- Occurs when both Ez and Bz are zero
- TEM Solution
  - Bz = 0 makes $\text{Curl}_t\ E_{TEM} = 0$
  - Ez = 0 makes $\text{Div}_t\ E_{TEM} = 0$
  - E_{TEM} is a solution to an electrostatic problem in 2D.
- Consequences
  - Propagates with k of plane wave in free space
  - B_{TEM} and E_{TEM} are related as in plane wave in free space
  - Can only exist with more than one conductor

Boundary Conditions

$$n \times E = 0$$

$$\Rightarrow E_z \,|_S = 0$$

$$n \cdot B = 0$$

$$\Rightarrow \frac{B_z}{\partial n} \,|_S = 0$$

- For a perfect electric conductor there are no fields inside and E_{tan} is continuous
- For a perfect electric conductor B is zero inside and B_{normal} is continuous

Waveguide Simplifications

$$\overline{H}_t = \frac{\pm 1}{Z} \hat{z} \times \overline{E}_t$$

$$Z_{TM} = \frac{k}{\omega \varepsilon} = \frac{k}{k_0}\sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_{TE} = \frac{\mu \omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\gamma^2 = \mu \varepsilon \omega^2 - k^2$$

$$\overline{E}_t = \pm \frac{ik}{\gamma^2} \nabla_t \overline{E}_z$$

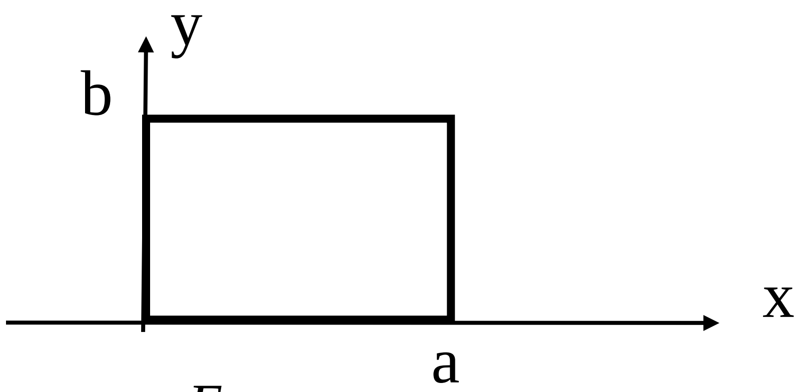
$$\overline{B}_t = \pm \frac{ik}{\gamma^2} \nabla_t \overline{B}_z$$

- First find Ez and Bz
- Then use gradient to find the associated transverse B or E
- Use impedance to find the associated transverse E and B
- Potentially there are 5 terms each for TE and TM but often there are only 3 terms each for the TE and TM

Generic Wave Properties

- The solution for the scalar function Ez or Bz introduces eigenvalues
- Phase propagation in the z direction only occurs once $\mu\varepsilon\omega^2$ exceeds the eigenvalue.
  - Cutoff frequency
  - Phase velocity
  - Group velocity
- Modes that are cutooff are said to be evanescent due to their exponential decay

Rectangular Waveguide Example (TM)



$$\psi = E_z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2 \right) \psi = 0$$

$$\psi \,|_S = 0$$

$$\overline{E}_t = \pm \frac{ik}{\gamma^2} \nabla_t \psi$$

$$\overline{H}_t = \frac{\pm 1}{Z} \hat{z} \times \overline{E}_t$$

$$Z_{TM} = \frac{k}{\omega \varepsilon} = \frac{k}{k_0} \sqrt{\frac{\mu}{\varepsilon}}$$

$$E_{zmn} = E_0 \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$$

$$\gamma_{mn}{}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

$$E_x = E_0 \frac{ik\pi}{\gamma_{mn}{}^2 a} \cos(\quad) \sin(\quad)$$

$$E_y = E_0 \frac{ik\pi}{\gamma_{mn}{}^2 b} \sin(\quad) \cos(\quad)$$

$$H_x = -E_0 \frac{ik\pi}{Z_{TM} \gamma_{mn}{}^2 b} \sin(\quad) \cos(\quad)$$

$$H_y = E_0 \frac{ik\pi}{Z_{TM} \gamma_{mn}{}^2 a} \cos(\quad) \sin(\quad)$$

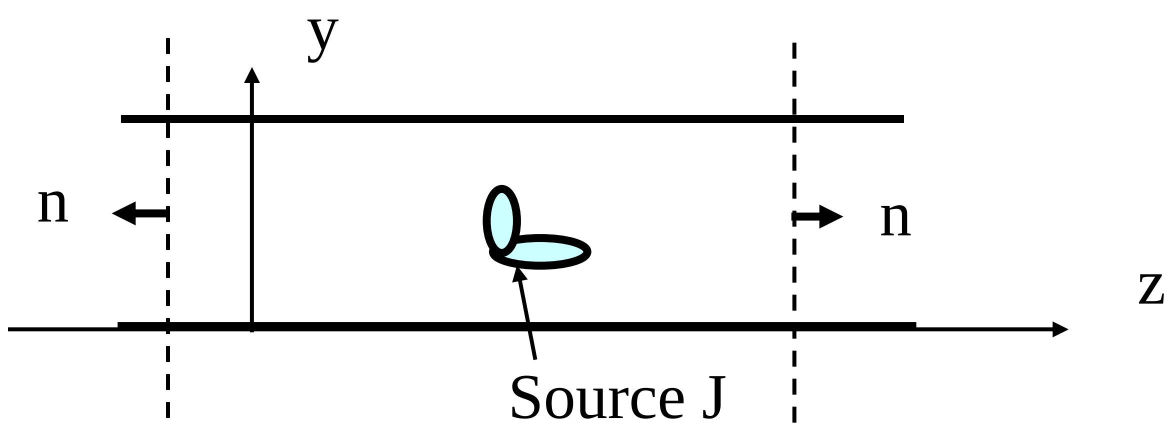
Rectangular Waveguide Orthonormal Modes

- The TE and TM modes are orthogonal to each other in any combination
- The TE and TM modes are orthogonal amongst them selves
- The can be normalized as has been don in Jackson 8.12

Representation of an Arbitrary Field

- The most general field in a source free region is a summation of TE and TM modes
- If the source is to the left then only fields propagating to the right need to be included
- FYI: Each mode by itself can be written as a sum of 4 planewaves
  - expand the sinusoidal behavior in x and y as exponentials
  - Compute the k-vector wave directions

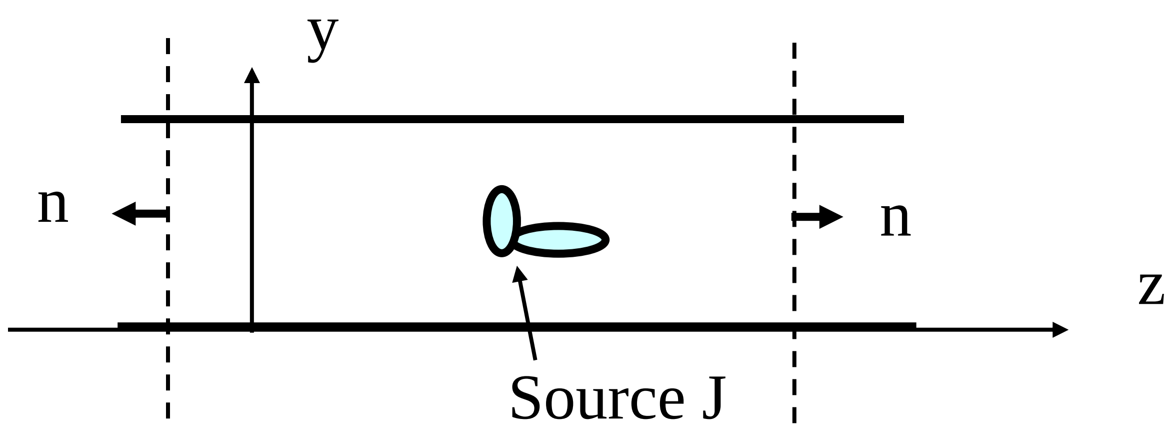
Fields Generated by a Localized Source



To find amplitude of a given mode propagating to the right

- General expansion for modes to right
- Assume the given mode of arbitrary amplitude is propagating to left from outside the boundary
- Apply reciprocity to the volume for these two sets of fields

Fields Generated by a Localized Source (Cont.)



Apply reciprocity to the volume for these two sets of fields

- Integral over wall is zero
- Integral over left cut-plane is zero as all modes going same direction
- Integral over right cut plane is proportional to outgoing mode amplitude times incoming mode amplitude
- Intergral over the source measures the component of the source with the x,y eigenfunction variation as well as phase coherence with z and is proportional to incoming mode amplitude
- Ratio cancels incoming amplitude and gives outgoing amplitude.