EE243 Advanced Electromagnetic Theory

Lec # 14: Sources in Guides

- **Solution Strategies**
- **Perfectly Electrically Conducting Guides**
- **Source Excitation via Reciprocity**
- **Plasmon fields and propagation constant**

Reading: Jackson 8.12

Plasmons now in Lecture 15

Overview

- Review construction of solutions
  - A component of A and F may be easier than Ez or Hz, but there is a little extra work to get Ez and Hz
- Reciprocity is useful for determining the excitation of a given waveguide mode.
- Waves on a single dielectric surface
  - Fields
  - Boundary conditions
  - Constraint
  - Plasmons for metals with $\varepsilon_2 < -\varepsilon_1$

Source Free Region

Harrington Strategy 3.12

$$\nabla^2 \overline{A} + k^2 \overline{A} = 0 \qquad \overline{E} = -\nabla \times \overline{F} + i\omega\mu\overline{A} + \frac{1}{i\omega\varepsilon}\nabla\left(\nabla \cdot \overline{A}\right)$$

$$\nabla^2 \overline{F} + k^2 \overline{F} = 0 \qquad \overline{H} = \nabla \times \overline{A} + i\omega\overline{F} + \frac{1}{i\omega\mu}\nabla\left(\nabla \cdot \overline{F}\right)$$

- With the Lorenz gauge A and F satisfy the wave equation and the fields are give by above equations.
- Choosing the vectors A and B to only be in the z direction is adequate.
- Each potentially contributes 5 components of the E,H combination.

Only Electric Potential in z Direction

$$\overline{A} = \psi \hat{z}$$

$$E_x = \frac{1}{-i\omega\varepsilon} \frac{\partial^2 \psi}{\partial x \partial z}$$

$$E_y = \frac{1}{-i\omega\varepsilon} \frac{\partial^2 \psi}{\partial y \partial z}$$

$$E_z = \frac{1}{-i\omega\varepsilon} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \psi$$

$$H_x = \frac{\partial \psi}{\partial y}$$

$$H_y = -\frac{\partial \psi}{\partial x}$$

$$H_z = 0$$

$$\overline{F} = \psi \hat{z}$$

$$E_x = -\frac{\partial \psi}{\partial y}$$

$$E_y = \frac{\partial \psi}{\partial x}$$

$$E_z = 0$$

$$H_x = \frac{1}{-i\omega\mu} \frac{\partial^2 \psi}{\partial x \partial z}$$

$$H_y = \frac{1}{-i\omega\mu} \frac{\partial^2 \psi}{\partial y \partial z}$$

$$H_z = \frac{1}{-i\omega\mu} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \psi$$

Jackson Strategy Eq. 8.26

$$\left[\nabla_t^2 + \left(\mu\varepsilon\omega^2 - k^2\right)\right]\overline{E} = 0$$

Jackson 8.2

$$\overline{E} = E_z\hat{z} + \overline{E}_t$$

$$\overline{E}_t = \frac{1}{\left(\mu\varepsilon\omega^2 - k^2\right)}\left[k\nabla_t E_z - \omega\hat{z}\times\nabla_t B_z\right]$$

$$\overline{B}_t = \frac{1}{\left(\mu\varepsilon\omega^2 - k^2\right)}\left[k\nabla_t B_z + \mu\varepsilon\omega\hat{z}\times\nabla_t E_z\right]$$

- E and B satisfy wave equation with transverse operator and $-k^2$
- Break up E and B into longitudinal and transverse
- Transverse fields can be found from E_z and B_z.

Comparison of Harrington and Jackson

- In a homogeneous source free region, it is adequate in general to independently solve for TWO SCALAR FUNCTIONS but which is easier
  - z components of vector potentials A for TM and F for TE, or
  - Ez and Hz themselves?
- The vector potential A (or F) in one direction is advantageous in finding the transverse components of H (or E)
  - Pair of zeros in taking the curl => easier to determine the boundary condition
  - Only propagating known information forward
  - But Ez (or Hz) requires mult. $(1/j\omega\varepsilon)(\omega^2\mu\varepsilon-k_z^2)$
- To be fair Jackson though, Eq. 8.26 only propagates known information forward once Ez and Hz are known
  - Ey could have been found from z cross grad Bz to get boundary condition that normal derivative of Bz must be zero for TE case.

Waveguide Simplifications (Revised)

$$\overline{E}_t = \pm \frac{ik}{\gamma^2} \nabla_t \overline{E}_z$$

$$\overline{B}_t = \pm \frac{ik}{\gamma^2} \nabla_t \overline{B}_z$$

$$\gamma^2 = \mu\varepsilon\omega^2 - k^2$$

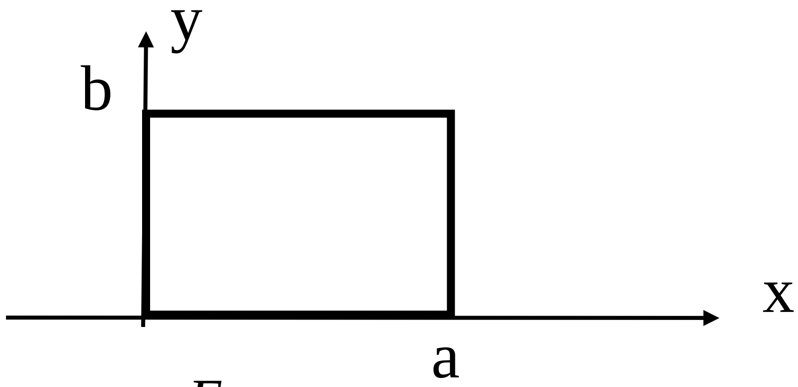
$$\overline{H}_t = \frac{\pm 1}{Z} \hat{z} \times \overline{E}_t$$

$$Z_{TM} = \frac{k}{\omega\varepsilon} = \frac{k}{k_0}\sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_{TE} = \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\varepsilon}}$$

- Set Boundary Condition
  - If TE Ez = 0 on p.e.c. sidewall.
  - If TM use 8.26 to get normal derivative of Bz = 0
- Solve for Ez and/or Bz
- Then use gradient to find the associated transverse B or E
- Use impedance to find the associated transverse E and B (or use 8.26)

Rectangular Waveguide Example (TM)



$$\psi = E_z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2 \right) \psi = 0$$

$$\psi \,|_S = 0$$

$$\overline{E}_t = \pm \frac{ik}{\gamma^2} \nabla_t \psi$$

$$\overline{H}_t = \frac{\pm 1}{Z} \hat{z} \times \overline{E}_t$$

$$Z_{TM} = \frac{k}{\omega \varepsilon} = \frac{k}{k_0} \sqrt{\frac{\mu}{\varepsilon}}$$

$$E_{zmn} = E_0 \sin(\frac{m \pi x}{a}) \sin(\frac{n \pi y}{b})$$

$$\gamma_{mn}{}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

$$E_x = E_0 \frac{ik\pi}{\gamma_{mn}{}^2 a} \cos(\ \ ) \sin(\ \ )$$

$$E_y = E_0 \frac{ik\pi}{\gamma_{mn}{}^2 b} \sin(\ \ ) \cos(\ \ )$$

$$H_x = -E_0 \frac{ik\pi}{Z_{TM} \gamma_{mn}{}^2 b} \sin(\ \ ) \cos(\ \ )$$

$$H_y = E_0 \frac{ik\pi}{Z_{TM} \gamma_{mn}{}^2 a} \cos(\ \ ) \sin(\ \ )$$

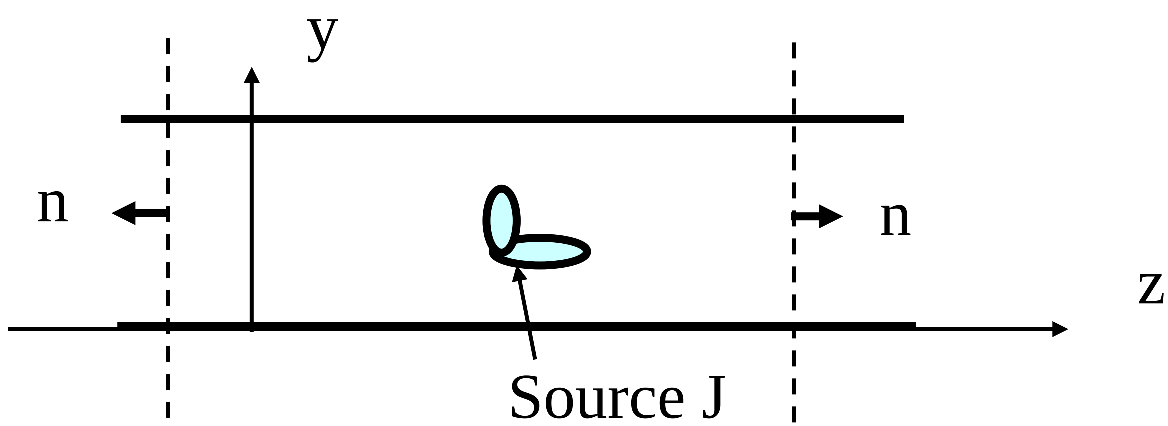
Generic Wave Properties

- The solution for the scalar function Ez or Bz introduces eigenvalues
- Phase propagation in the z direction only occurs once $\mu\varepsilon\omega^2$ exceeds the eigenvalue.
  - Cutoff frequency
  - Phase velocity
  - Group velocity
- Modes that are cutoff are said to be evanescent due to their exponential decay

Rectangular Waveguide Orthonormal Modes

- The TE and TM modes are orthogonal to each other in any combination
- The TE and TM modes are orthogonal amongst them selves
- The waves can be normalized as has been done in Jackson 8.12
- The most general field in a source free region is a summation of TE and TM modes

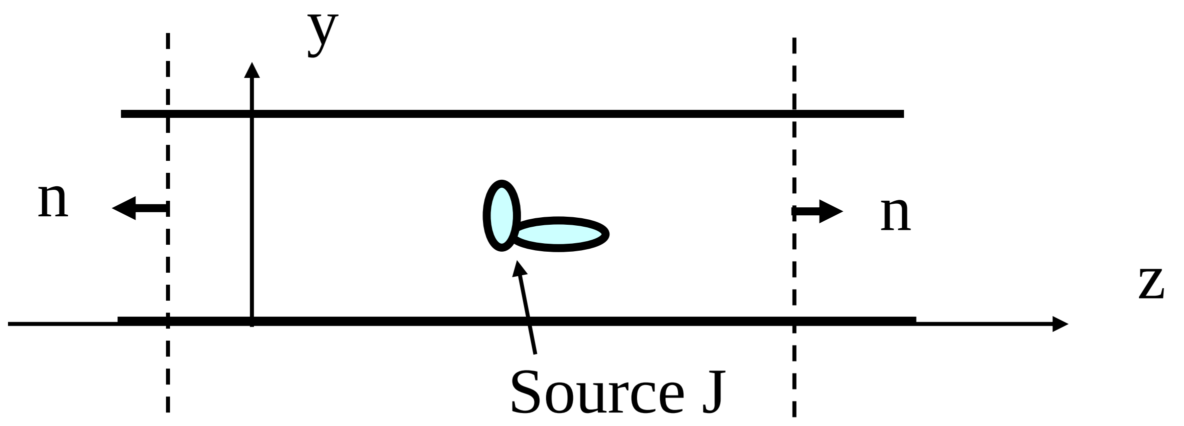
Fields Generated by a Localized Source



To find amplitude of a given mode propagating to the right

- General expansion for modes to right
- Assume the given mode of arbitrary amplitude is propagating to left from outside the boundary
- Apply reciprocity to the volume for these two sets of fields

Fields Generated by a Localized Source (Cont.)



Apply reciprocity to the volume for these two sets of fields

- Integral over wall is zero
- Integral over left cut-plane is zero as all modes going same direction
- Integral over right cut plane is proportional to outgoing mode amplitude times incoming mode amplitude
- Intergral over the source measures the component of the source with the x,y eigenfunction variation as well as phase coherence with z and is proportional to incoming mode amplitude
- Ratio cancels incoming amplitude and gives outgoing amplitude.

Representation of Fields in Guide

$$\overline{E}^{+} = \sum_{\lambda} A_{\lambda}^{+}\left[\overline{E}_{t\lambda}(x,y) + \overline{E}_{z\lambda}(x,y)\right] e^{-ik_{\lambda}z}$$

$$\overline{H}^{+} = \sum_{\lambda} A_{\lambda}^{+}\left[\overline{H}_{t\lambda}(x,y) + \overline{H}_{z\lambda}(x,y)\right] e^{-ik_{\lambda}z}$$

$$\overline{E}^{-} = \sum_{\lambda} A_{\lambda}^{-}\left[\overline{E}_{t\lambda}(x,y) - \overline{E}_{z\lambda}(x,y)\right] e^{+ik_{\lambda}z}$$

$$\overline{H}^{-} = \sum_{\lambda} A_{\lambda}^{-}\left[-\overline{H}_{t\lambda}(x,y) + \overline{H}_{z\lambda}(x,y)\right] e^{+ik_{\lambda}z}$$

$$\overline{E}^{-}_{TEST} = C_{\lambda}^{-}\left[\overline{E}_{t\lambda}(x,y) - \overline{E}_{z\lambda}(x,y)\right] e^{+ik_{\lambda}z}$$

$$\overline{H}^{-}_{TEST} = C_{\lambda}^{-}\left[-\overline{H}_{t\lambda}(x,y) + \overline{H}_{z\lambda}(x,y)\right] e^{+ik_{\lambda}z}$$

- Localized source J creates waves
  - Index λ goes over TE, TM, m, n
  - To right of source only waves to +z and sum over all TE and TM modes that propagate
  - To left of source only waves to –z and sum over all TE and TM waves
  - To left fields have signs altered div E = div H = 0
  - Test wave from outside going to left across volume

Apply Reciprocity Formulation

$$\nabla \cdot \left(\overline{E}_{TEST} \times \overline{H}_{\lambda}^{\pm} - \overline{E}_{\lambda}^{\pm} \times H_{TEST} \right) = \overline{J} \cdot \overline{E}_{TEST}$$

$$\int_S \left(\overline{E}_{TEST} \times \overline{H}_{\lambda}^{\pm} - \overline{E}_{\lambda}^{\pm} \times H_{TEST} \right) \cdot \hat{n} da = \int_V \overline{J} \cdot \overline{E}_{TEST} d^3x$$

- Source J produces the modes leaving the localized source region with amplitudes A_λ
- Source free TEST wave enters the volume and takes a measure of E
- Take Poynting Theorem like interaction

Evaluating Terms on Surfaces

$$\left(\overline{E}_{TEST} \times \overline{H}_{\lambda}^{\pm} - \overline{E}_{\lambda}^{\pm} \times H_{TEST}\right) \cdot \hat{n}$$

$$\left(C\overline{E}_{\tau}^{-} \times A_{\tau}\overline{H}_{\tau}^{-} - A_{\tau}\overline{E}_{\tau}^{-} \times CH_{\tau}^{-}\right) \cdot \left(-\hat{z}\right) = 0$$

$$\left(C\overline{E}_{\tau}^{-} \times A_{\tau}\overline{H}_{\tau}^{+} - A_{\tau}\overline{E}_{\tau}^{+} \times CH_{\tau}^{-}\right) \cdot \hat{z}$$

$$= \left(C\overline{E}_{\tau}^{+} \times A_{\tau}\overline{H}_{\tau}^{+} - A_{\tau}\overline{E}_{\tau}^{+} \times \left(-CH_{\tau}^{+}\right)\right) \cdot \hat{z}$$

$$= 2CA_{\tau}(\overline{E}_{\tau}^{+} \times \overline{H}_{\tau}^{+}) \cdot \hat{z} = 2CA_{\tau}Power$$

- Integral over p.e.c. is zero because E parallel to the surface is zero.
- Intergral over left plane
  - Due to otrhogonality between TE and TM and within TE or TM this integral could only be non-zero for λ = TEST = τ.
  - But for this term the two cross products are identical and cancel

Combining to find the mode amplitude

$$\int_S \left(\overline{E}_{TEST} \times \overline{H}_\lambda^{\pm} - \overline{E}_\lambda^{\pm} \times H_{TEST}\right) \cdot \hat{n} da = \int_V \overline{J} \cdot \overline{E}_{TEST} d^3x$$

$$2CA_\tau Power = \int_V \overline{J} \cdot C\overline{E}_\tau^{-} d^3x$$

$$A_\tau = \frac{1}{2Power} \int_V \overline{J} \cdot \overline{E}_\tau^{-} d^3x$$

- The amplitude of the test wave cancels
- The mode amplitude is half of the power normalized integral of the electric field of the inward traveling mode with the source current

Plasmons now in Lecture 15