EE243 Advanced Electromagnetic Theory

Fixed Slides 5, 9

Lec # 20 Coupled Mode Theory (Cont. 2)

- **Coupled Mode Eigenvalues and Eigenfunctions**
  - **Modes Lock Together as Super Modes**
- **Leaky Waves on Structures with Radiation Loss**
- **Mode Crossings in $\omega-\beta$ diagrams: two types**
- **Bloch Waves (Propagation and Stop Bands)**
- **Floquet Theorem for Periodic Structures**
- **$\omega-\beta$ Diagrams for Periodic Structures**

Reading: Haus 7.6, 8.1, (9,1-9.2 lite), Tamir 3.1.4, Collin 4.8-4.9, 8.1, 8.2, 8.6, 8.8, 5.7-5.8

Overview

Implications and generalizations of coupled modes.

- Super modes are locked sets of modes with new k-vectors
- Leaky modes have radiation loss, hence complex k-vectors
- Relative direction of the group and phase velocity determines phase or attenuation type interaction

Generalizations for Periodic Structures

- Bloch Waves (Allowed Crystal Super Modes)
- Floquet Most General Representation of a periodic field
- $\omega-b$ Diagrams for periodic structures with k-vector combs and intersections

Coupled Modes as Eigenfunction Problem

Use to check Kogelnik Solution in Eq. 2.6.30-31.

$$\overline{X}_A = \begin{Bmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{Bmatrix}$$

$$\overline{X}'_A = \overline{\overline{M}} \cdot \overline{X}_A$$

$$\overline{X}'_{ei} = -j\lambda_i \overline{X}'_{ei}$$

$$0 = \left[\overline{\overline{M}} - j\lambda_i \overline{\overline{I}}\right] \cdot \overline{X}_{ei}$$

$$Det\left[\overline{\overline{M}} - j\lambda_i \overline{\overline{I}}\right] = 0$$

- Construct a vector of mode amplitudes
- Rate equation can be written as derivative of mode vector equal to a coupling matrix M times mode vector
- Look for source free solutions (eigenvalues) by substituting an arbitrary exponential variation
- Determinant constrains arbitrary exponential (eigenvalues)

Coupled Modes as Eigenfunction Problem (Cont.)

$$\lambda_i^2 - \delta^2 - k^2 = 0$$

$$\lambda_i = \pm\sqrt{\delta^2 + k^2}$$

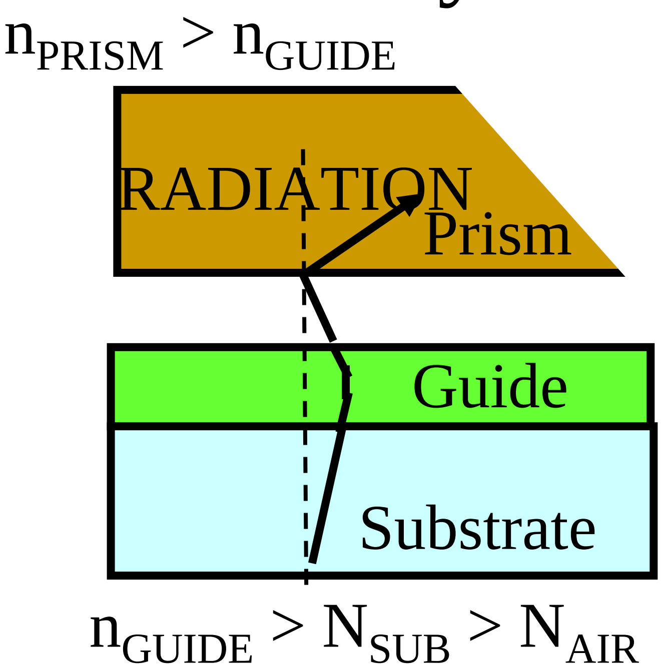
$$\left(M_{21} + j\lambda_1\right)X_{e1,1} + M_{12}X_{e1,2} = 0$$

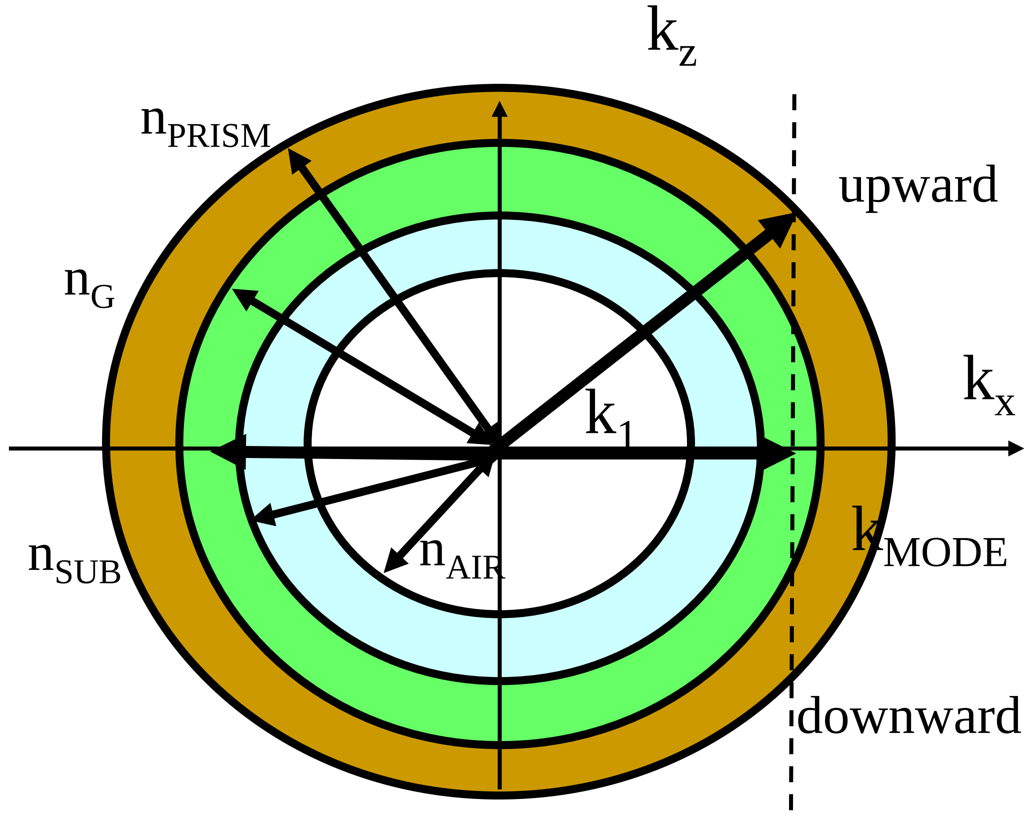
$$M_{21}X_{e2,1} + \left(M_{22} + j\lambda_2\right)X_{e2,2} = 0$$

$$\overline{X}_A = \sum_i \overline{X}_{ei} e^{-j\lambda_i z}$$

- Eigenfunctions are found by back substituting eigenvalues
- These eigenfunctions are the Super Modes and show the field behavior in the cross section. (It changes as ω changes)
- Homogeneous solution is sum over eigenfunctions with their eigenvalue z dependence plus boundary condition at z locations
- A solution driven by an imposed (forced) z variation will take on that z-variation with eigenfunctions added to match z transition conditions. (Like a circuit - forced time variation and transient.)

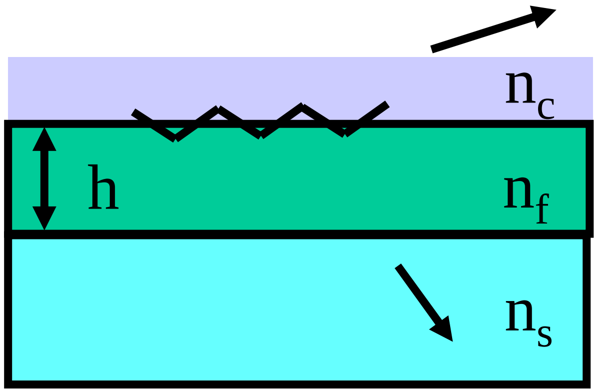
Leaky Waves: Uniform Structure

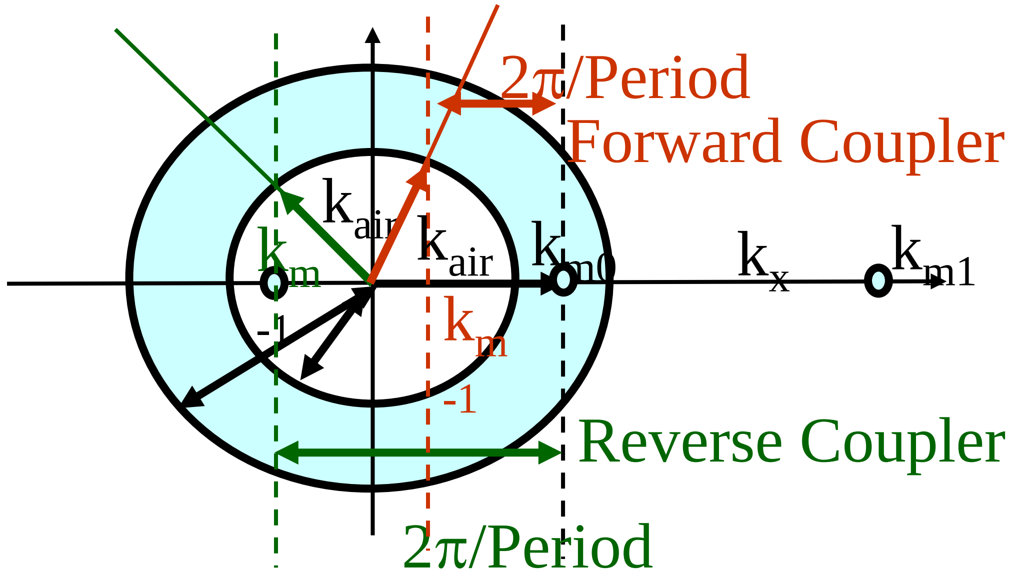




- When ever modes can transfer part of their energy to radiation they are termed Leaky Waves and their k-vectors parallel to the propagation direction are complex.
- This can occur by tunneling into the high n region (Prism Coupler)
- This complex z variation can occur by giving or receiving energy

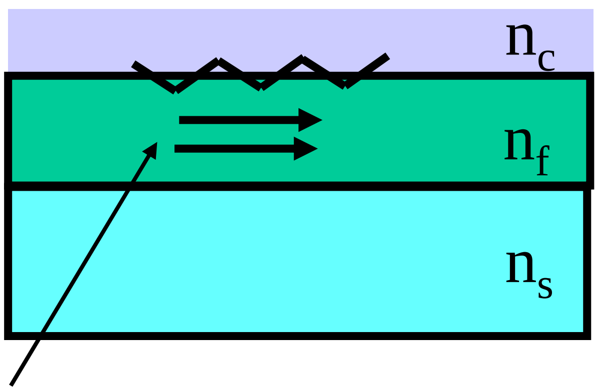
Leaky Waves: Periodic Structure



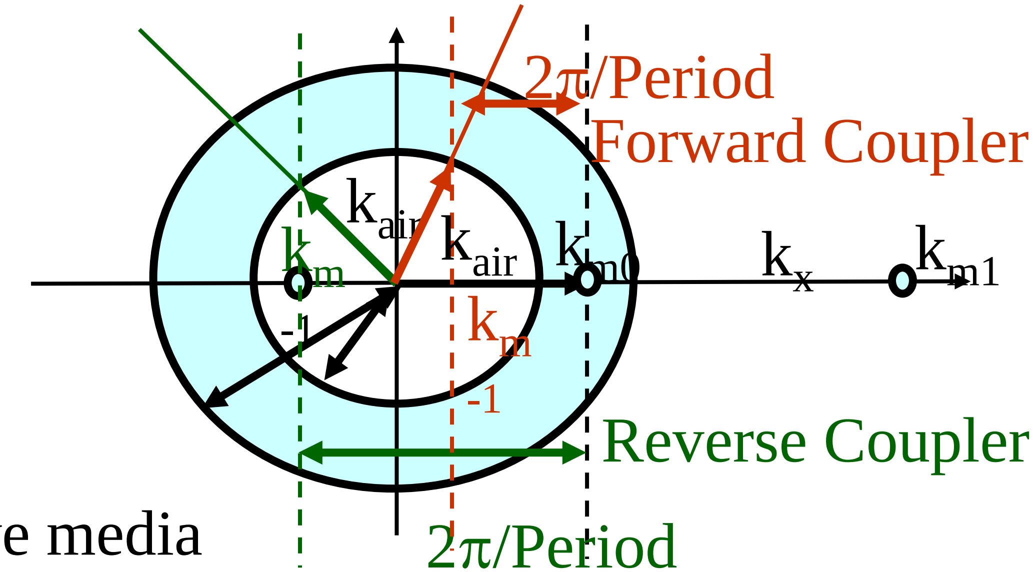


- When ever modes can transfer part of their energy to radiation they are termed Leaky Waves and their k-vectors parallel to the propagation direction are complex.
- A periodic geometrical variation can produce a k-vector that radiates.
- The radiation loss requires an associated attenuation

Wave Media Interaction



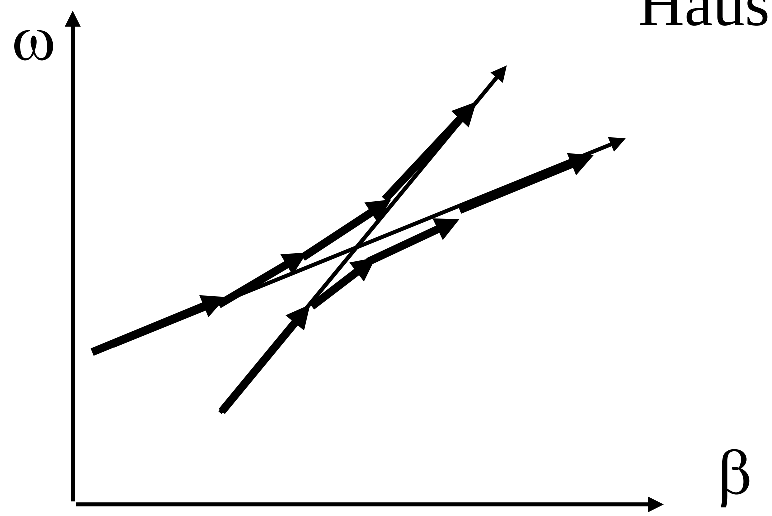
Flowing or standing wave media



- Standing waves can produce periodic media modulation
  - Acousto-Optical Modulator (slow moving periodic structure)
- Flowing carriers can add and remove energy
  - Traveling-wave tubes with electron beams
- Nonlinear media effects can couple modes
  - Modelocking

Coupled Mode: v_g and v_p Same Direction

Haus 7.6

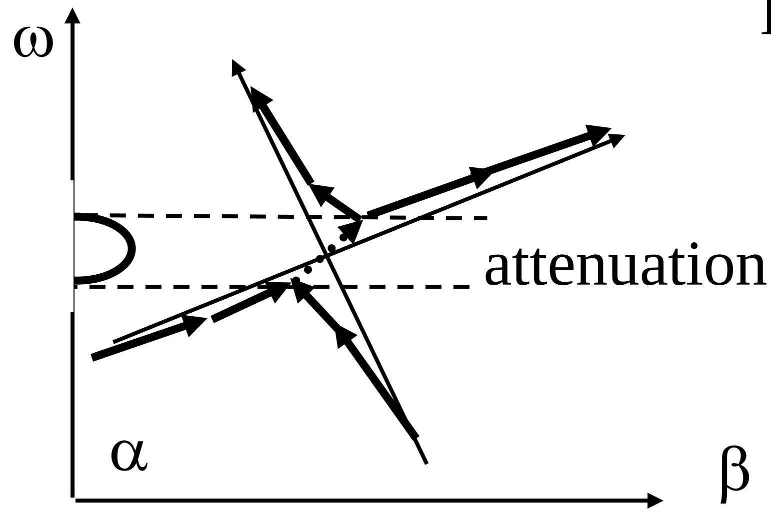


When the group and phase velocities are in the same direction

- The eigenvalues (β's) move away from each other
- The displacement is proportional to the coupling coefficient
- The eigenfunctions (Super Modes) associated with eigenvalue (β) continuously change identity in passing through the crossing point

Coupled Mode: v_g and v_p Opposite Direction

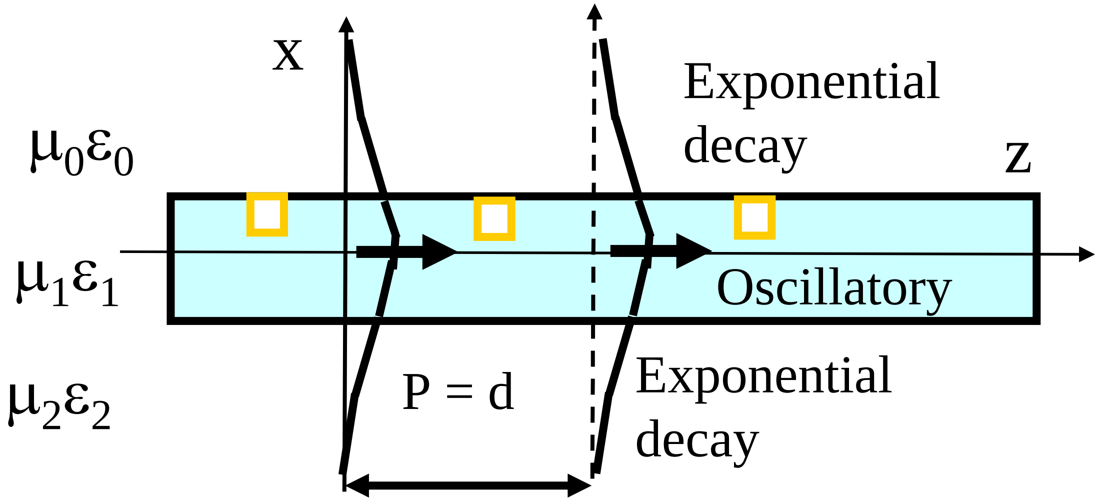
Haus 7.6



When the group and phase velocities are in the opposite direction

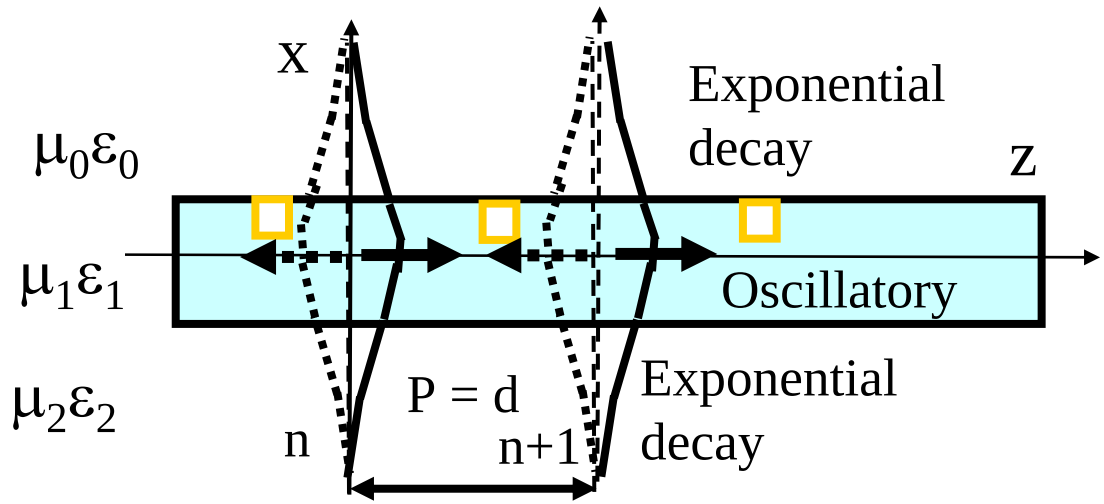
- The eigenvalues (β's) move toward each other and merge
- After they merge an attenuation region appears
- The region over which they merge and the level of attenuation is proportional to the coupling coefficient

Periodic Media Complications and Fixes



- Coupled Mode Theory
  - Describes the change with distance when coupling is introduced
  - Each eigenvalue and eigenvector gives a Super Mode distribution and its β
- For a periodic structure
  - The coupling is not uniform with distance
  - The fields are not uniform with distance
  - A Super Mode becomes a sum over an infinite number of periodic k-vectors
  - But it is possible to compare fields at z values that differ by the period P = d

Bloch Wave Concept and Constraint



- At each cut plane n and n+1
  - Super Mode to right c_n^+ and c_{n+1}^+
  - Super Mode to the left c_n^- and c_{n+1}^-
- Require periodic behavior $e^{-\gamma d}$ between cut planes
  - $c_{n+1}^+ = c_n^+ e^{-\gamma d}$
  - $c_{n+1}^- = c_n^- e^{-\gamma d}$
- Integrate coupling coefficients over period (A matrix)
  - Aij = integrate (Super Mode)$_i$ $\Delta\varepsilon$ (Super Mode)$_j$

Bloch Wave Constraint and Matrix

Collin 8.2 Eq 8.19-21

$$\begin{bmatrix} c_{n+1}^{+} \\ c_{n+1}^{+} \end{bmatrix} = e^{-\gamma d} \begin{bmatrix} c_{n}^{+} \\ c_{n1}^{+} \end{bmatrix}$$

- Bloch Constraint of simple complex factor

$$\begin{bmatrix} c_{n}^{+} \\ c_{n1}^{+} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_{n+1}^{+} \\ c_{n+1}^{+} \end{bmatrix}$$

- Per period change described by the A matrix which has elements similar to $K_{i,j}$

$$\begin{bmatrix} A_{11} - e^{-\gamma d} & A_{12} \\ A_{21} & A_{22} - e^{-\gamma d} \end{bmatrix} \begin{bmatrix} c_{n+1}^{+} \\ c_{n+1}^{+} \end{bmatrix} = 0$$

- Homogeneous Constraint Determinant = 0

$$\cosh(\gamma d) = \frac{A_{11} + A_{22}}{2}$$

- Constraint on γ gives waves allowed in periodic structure

Relationship of A-Parameters to S-Parameters

Collin 4.9 Eq 4.80

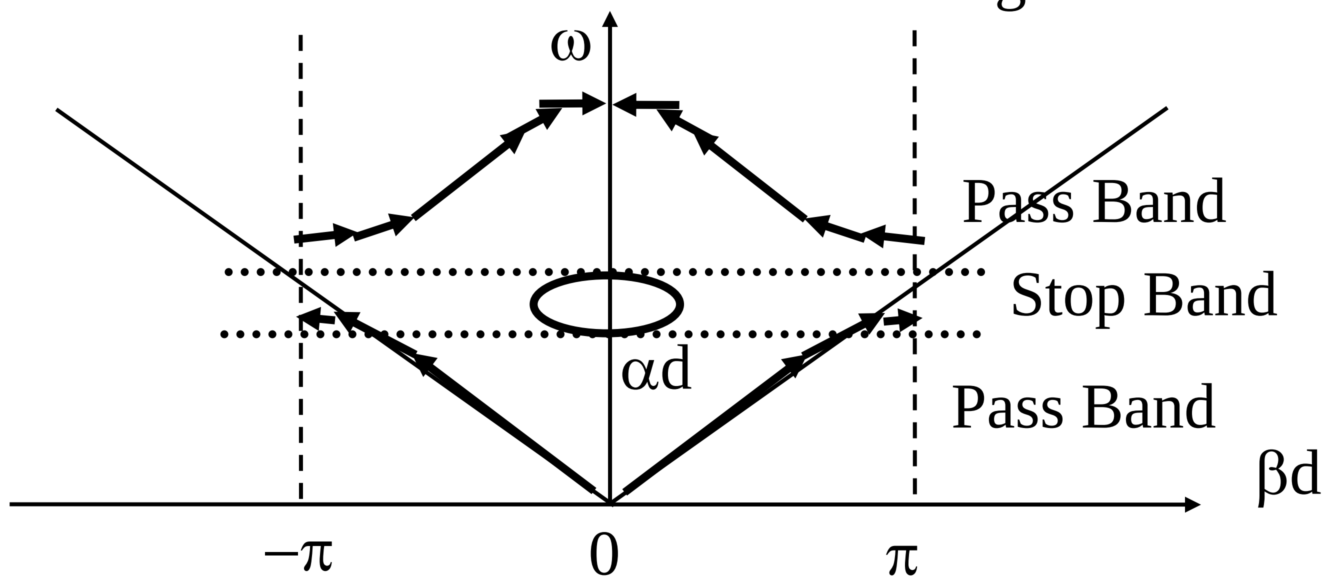
$$\begin{bmatrix} c_n^+ \\ c_{n1}^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_{n+1}^+ \\ c_{n+1}^+ \end{bmatrix}$$

$$\begin{bmatrix} c_n^+ \\ c_{n1}^+ \end{bmatrix} = \begin{bmatrix} 1/S_{12} & 1S_{22}/S_{12} \\ S_{11}/S_{12} & \left(S_{12}^2 - S_{11}S_{22}\right)/S_{12} \end{bmatrix} \begin{bmatrix} c_{n+1}^+ \\ c_{n+1}^+ \end{bmatrix}$$

- Could work out each term by taking ratios
- Determinant = of A matrix is unity

Bloch Wave ω–β Diagram

Collin Fig. 8.8



- Example from Collin of a coaxial line with periodic rings
- When frequency increases through the region where the rings become near λ/2 in spacing
  - The group velocity drops to zero
  - A pure attenuation region is encountered
  - And propagation then resumes starting from a zero group velocity

Floquet Expansion for Fields (x,y,z)

Collin 8.8 Eq 8.50-52

$$\overline{E}_p(x,y,z) = \sum_{n=-\infty}^{\infty} \overline{E}_{pn}(x,y) e^{-j2n\pi z/d}$$

$$\overline{E}_{pn}(x,y) = \frac{1}{d}\int_0^d \overline{E}_p(x,y,z) e^{j2n\pi z/d}$$

$$\overline{E}_p(x,y,z) = \sum_{n=-\infty}^{\infty} \overline{E}_{pn}(x,y) e^{-j\beta_n z}$$

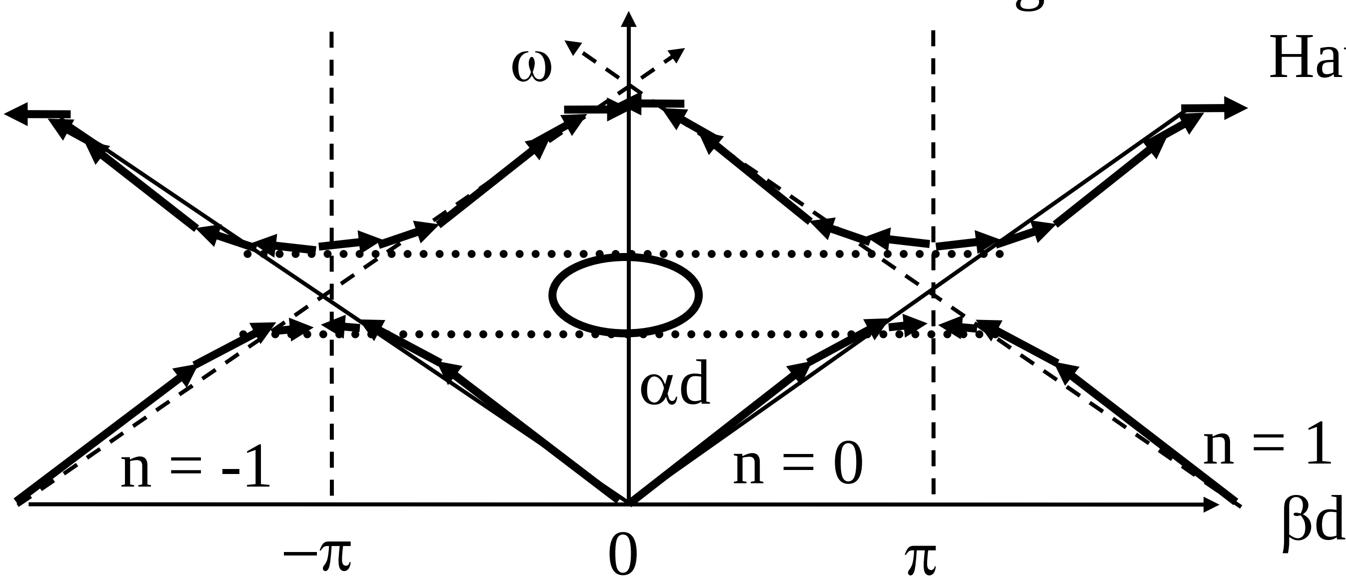
$$\gamma = j\beta$$

$$\beta_n = \beta + 2n\pi / d$$

- A periodic function can be expanded in complex Fourier series in z
- The common behavior $e^{-\gamma d}$ between periodic cut planes can be factored out

ω–β Diagram for a Periodic Structure

Collin Fig. 8.8 Generalized
Haus 8.1



- This plot shows the ω–β behavior of each term in the Floquet expansion
- When any term is nearly phase matched to another term the interaction is enhanced and a noticeable perturbation takes place.
- Note that while each spatial harmonic has its own phase variation they each have the same group velocity.