EE243 Advanced Electromagnetic Theory

Lec # 22 Scattering and Diffraction

- **Scattering From Small Objects**
- **Scattering by Small Dielectric and Metallic Spheres**
- **Collection of Scatters**
- **Spherical Wave Expansions**
  - **Scalar**
  - **Vector**

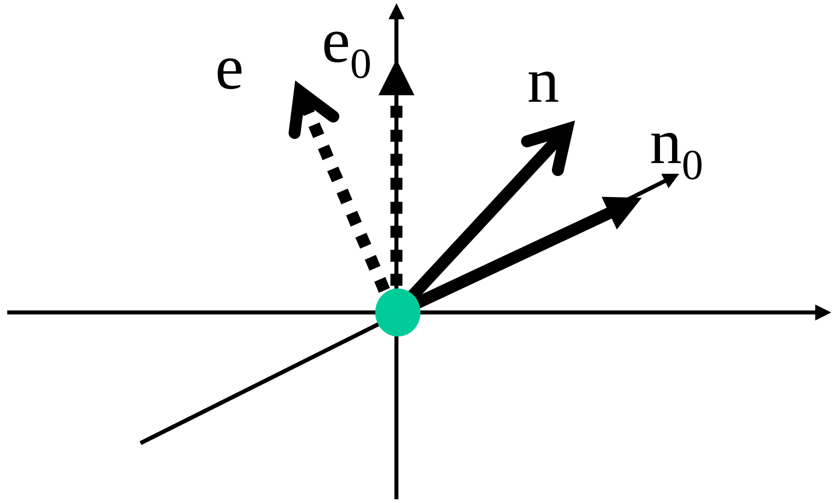
Reading: Jackson Chapter 10.1, 10.3,
lite on both 10.2 and 10.4

Overview

Scattering is similar to radiation but often requires simultaneously modeling the creation of polarization and currents from stimulation by an external source.

- Small scatterers are treated by dipole moments.
- Intermediate scatterers require expansion in many spherical harmonics.
- Large scatterers can be treated by approximation in various scalar and vectod diffraction integrals

Scattering by Dipoles Induced in Small Scatterers



- Incident field is in direction $\mathbf{n_0}$ and has polarization $\mathbf{e_0}$
- They induce electric and magnetic dipole moments
- Scattered field is in direction **n** and has polarization **e**
- Note that for the far field there are two choices for each of $\mathbf{e_0}$ and **e** but one choice relative to the plane formed by $\mathbf{n_0}$ and **n**

Scattering by Dipoles Induced in Small Scatterers

Jackson 10.1.A

$$\overline{E}_{inc} = e_0 E_0 e^{ik\hat{n}_0 \cdot \overline{x}}$$

$$\overline{H}_{inc} = \hat{n}_0 \times \overline{E}_{inc} / Z_0$$

$$\overline{p} = induced\_electric\_dipole$$

$$\overline{m} = induced\_magnetic\_dipole$$

$$\overline{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} \left[(\hat{n} \times \overline{p}) \times n - n \times \overline{m} / c \right]$$

$$\overline{H}_{sc} = \hat{n} \times \overline{E}_{sc} / Z_0$$

- Incident fields induce electric and magnetic dipole moments
- Far fields from are then found from these moments

Differential Scattering Cross Section

$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{e};\hat{n}_0,\hat{e}_0\right)=\frac{r^2\,\dfrac{1}{2Z_0}\left|\hat{e}^*\cdot\overline{E}_{sc}\right|^2}{\dfrac{1}{2Z_0}\left|\hat{e}_0{}^*\cdot\overline{E}_{inc}\right|^2}$$

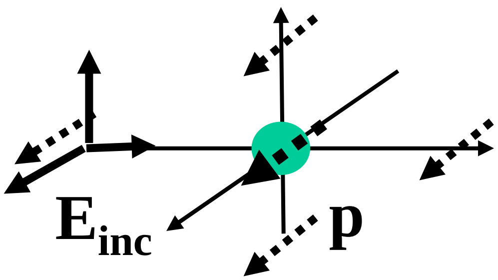
$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{e};\hat{n}_0,\hat{e}_0\right)=\frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{e}^*\cdot\overline{p}+\left(\hat{n}\times\hat{e}^*\right)\cdot\overline{m}/c\right|^2$$

- **n** is in observation direction with polarization **e**, while incident flux is in direction $\mathbf{n_0}$ with polarization $\mathbf{e_0}$.
- Defined as the outgoing power radiated per unit solid angle divided by the incident power per unit area. It is related to the bistatic cross section.
- Then specialize to the case of the electric and magnetic dipole moments of small scatterers.
- Integrating over both polarizations and all angles gives the effective area of the scatterer

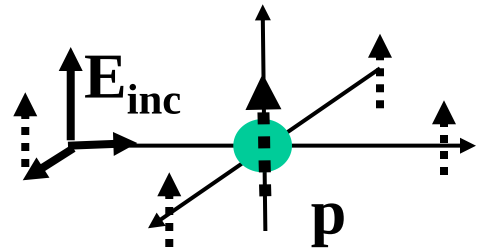
Scattering from a Small Dielectric Sphere

$$\overline{p} = 4\pi\varepsilon_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) a^3 \overline{E}_{inc}$$

Jackson 10.1.B



$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}; \hat{n}_0, \hat{e}_0) = k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \left| \hat{e}^* \cdot \hat{e}_0 \right|^2$$



$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2$$

- Dipole **p** is in the direction of the incident field and equal to the static polarization (same weight factor and proportional to volume).
- Radiation is proportional the observation polarization direction dotted with the incident polarization. This gives cosθ in one angle and constant in ϕ.
- Strength is 6-th power of size (volume squared) and 4-th power relative to size in wavelengths. (This explains the creation of the blue sky success of horizontally polarized sun glasses).
- Strongest and equal in forward and backward directions.

Scattering from a Small p.e.c. Sphere

Jackson 10.1.C

$$\overline{p} = 4\pi\varepsilon_0 a^3 \overline{E}_{inc}$$

$$\overline{m} = -2\pi a^3 \overline{H}_{inc}$$

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{e};\hat{n}_0,\hat{e}_0) = k^4 a^6 \left| \hat{e}^* \cdot \hat{e}_0 - \frac{1}{2}(\hat{n}\times\hat{e}^*)\cdot(\hat{n}_0\times\hat{e}_0) \right|^2$$

p and **m**

- Both exist
- Are at right angles
- Interfere coherently
  - produce a + b cosθ type patterns
  - low forward (1/3) and high backward (2x) scattering

Collection of Scatterers

$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{e};\hat{n}_0,\hat{e}_0\right)=\frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\sum_j\left[\hat{e}^*\cdot\overline{p}_j+\left(\hat{n}\times\hat{e}^*\right)\cdot\overline{m}_j/c\right]e^{i\overline{q}\cdot\overline{x}_j}\right|^2$$

$$\overline{q}=k\hat{n}_0-k\hat{n}$$

Jackson 10.1.D

$$F\left(\overline{q}\right)=\left|\sum_j e^{i\overline{q}\cdot\overline{x}_j}\right|^2$$

$$F\left(\overline{q}\right)=\sum_j\sum_i e^{i\overline{q}\cdot\left(\overline{x}_j-\overline{x}_i\right)}$$

- Assume p and m corrected for being inside media
- Sum over all scatterers including relative phase measured with respect to incident direction $\mathbf{n_0}$ and scattered direction **n**
- F(q) is N (number of scatterers) in forward direction and drops quickly to zero except for crystal structures with Bragg effect.
- Can be used to measure range of intermolecular forces that produce density fluxuations (critical opalescence).

Scalar Spherical Wave Representation

$$\Psi(\bar{x},\omega)=\sum_{l,m} f_{lm}(r)Y_{lm}(\theta,\phi)$$

Jackson 10.1.D

$$\Psi(\bar{x},\omega)=\sum_{l,m}\left[A_{lm}^{(1)}h_l^{(1)}(kr)+A_{lm}^{(2)}h_l^{(2)}(kr)\right]Y_{lm}(\theta,\phi)$$

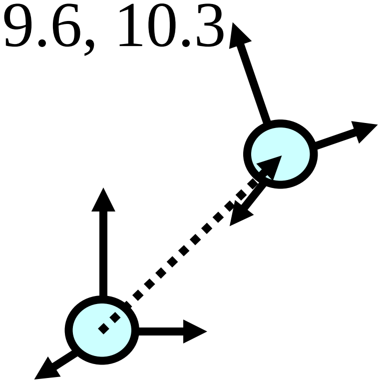
$$h_l^{(1)}(kr)\rightarrow\left(-i^{l+1}\frac{e^{ikr}}{kr}\right)$$

$$h_l^{(2)}(kr)=\left[h_l^{(2)}(kr)\right]^*$$

- Solution to scalar wave equation
- Spherical Harmonics $Y_{lm}(\theta,\phi)$
- Radial variation depends only on index l
  - Match boundary conditions on surface(s) at fixed r

Scalar Spherical Wave Representation: Examples

Jackson 9.6, 10.3



$$\frac{e^{ik|\bar{x}-\bar{x}'|}}{4\pi|\bar{x}-\bar{x}'|} = ik\sum_l J_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

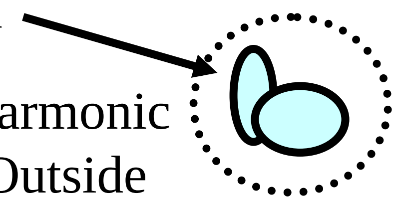
$$e^{i\bar{k}\cdot\bar{x}} = ik\sum_l i^l J_l(kr)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$e^{i\bar{k}\cdot\bar{x}} = \sum_{l=0}^{\infty} i^{-l}\sqrt{4\pi(2l+1)}J_l(kr)Y_{l0}(\gamma)$$

$h_l^{(1)}(kr)$

Numerical

Spherical Harmonic Expansion Outside



- Scalar Green's Function
- Plane wave in two forms
- Replace Numerical Grid outside object (Mei Method)
- Translation/rotation in coordinate systems
  - Addition Theorem for Spherical Harmonics
  - Sphere-Sphere interactions

Vector Spherical Wave Representation

Jackson 10.3

$$\bar{L} = \frac{1}{i}\left(\bar{r} \times \nabla\right)$$

$$\bar{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\bar{L}Y_{lm}(\theta,\phi)$$

$$\bar{H} = \sum_{l,m}\left[a_E(l,m)f_l(kr)\bar{X}_{lm} - \frac{i}{k}a_M(l,m)\nabla \times g_l(kr)\bar{X}_{lm}\right]$$

$$\bar{E} = Z_0\sum_{l,m}\left[\frac{1}{k}a_E(l,m)\nabla \times f_l(kr)\bar{X}_{lm} + a_M(l,m)g_l(kr)\bar{X}_{lm}\right]$$

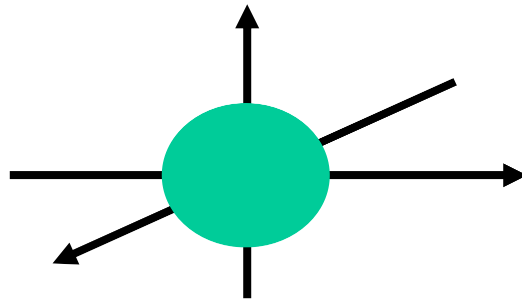
$$g_l(kr) = A_l^{(1)}h_l^{(1)}(kr) + A_l^{(2)}h_l^{(2)}(kr)$$

$$f_l(kr) = B_l^{(1)}h_l^{(1)}(kr) + B_l^{(2)}h_l^{(2)}(kr)$$

- Operator **L** gives compact notation
- Expressed in terms of electric and magnetic multipoles
- Source and Boundary conditions on surface at fixed r
  - Radial E and H component on a Sphere are adequate

Vector Spherical Wave Representation: Example

Jackson 10.4



- Total field sum of incident and scattered
- Expand scattered field
  - Outgoing waves (only) outside
  - Both types but no incidint field inside
  - Rotationally symmetric m = +1 and -1 (only)
- Boundary conditions
  - E_{tan} and H_{tan} continuous on boundary of sphere
- Tractable for
  - Conducting Sphere (Mie)
  - Dielectric Sphere ?
- Useful for checking numerical simulators