EE 221a Homework 9 Solutions [1]

Fall 2007

Problem 1. Starting from the individual state equations:

$$
\begin{aligned}
\dot{x_1} &= A_1 x_1 + B_1 e_1 \\
y_1 &= C_1 x_1 + D_1 e_1 \\
\\
\dot{x_2} &= A_2 x_2 + B_2 e_2 \\
y_2 &= C_2 x_2 \\
\\
e_1 &= u_1 - y_2 \\
e_2 &= u_2 + y_1
\end{aligned}
\tag{1}
$$

we obtain the state equations for the composite system:

$$
\begin{aligned}
\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} &= \begin{pmatrix} A_1 & -B_1 C_2 \\ B_2 C_1 & A_2 - B_2 D_1 C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ B_2 D_1 & B_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} C_1 & -D_1 C_2 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\end{aligned}
\tag{2}
$$

To compute conditions for stabilizability, we must compute

$$
\begin{aligned}
&\operatorname{rank} \begin{pmatrix} sI - A_1 & B_1 C_2 & B_1 & 0 \\ -B_2 C_1 & sI - A_2 + B_2 D_1 C_2 & B_2 D_1 & B_2 \end{pmatrix} \\
&= \operatorname{rank} \begin{pmatrix} sI - A_1 & 0 & B_1 & 0 \\ -B_2 C_1 & sI - A_2 & B_2 D_1 & B_2 \end{pmatrix} \\
&= \operatorname{rank} \begin{pmatrix} sI - A_1 & 0 & B_1 & 0 \\ 0 & sI - A_2 & 0 & B_2 \end{pmatrix}
\end{aligned}
\tag{3}
$$

where the first equality was obtained by post multiplying the col. block 3 by $-C_2$ and adding the result to col. block 2. The second equality was obtained by firs post multiplying col. block 4 by $-D_1$ and adding the result to col. block 3, then post multiplying col. block 4 by C_1 and adding the result to col. block 1.

Because of the form of the final matrix, we need only consider the cases when s is equal to an eigenvalue of either A_1 or A_2 – the matrix will not drop rank for other values of s. Further, since S_1 and S_2 are stabilizable, the matrix has $2n$ LI rows for $s = \lambda$ where λ is an unstable eigenvalue of either A_1 or A_2. Therefore, the composite system is stabilizable.

[1] Thanks to Onureena Banerjee for providing part of the solution

Checking detectability is similar: compute

$$\operatorname{rank}\begin{pmatrix} sI - A_1 & B_1C_2 \\ -B_2C_1 & sI - A_2 + B_2D_1C_2 \\ C_1 & -D_1C_2 \\ 0 & C_2 \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} sI - A_1 & B_1C_2 \\ 0 & sI - A_2 \\ C_1 & -D_1C_2 \\ 0 & C_2 \end{pmatrix} \tag{4}$$

$$= \operatorname{rank}\begin{pmatrix} sI - A_1 & 0 \\ 0 & sI - A_2 \\ C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

where we obtain the first equality by pre-multiplying row block 3 by B_2 and adding the result to row block 2. We obtained the second equality by first pre-multiplying row block 4 by $-B_1$ and adding the result to row block 1, and then pre-multiplying row block 4 by D_1 and adding the result to row block 3.

Again, due to the form of this final matrix, we need only consider the cases where s is equal to an eigenvalue of either A_1 or A_2. Since S_1 and S_2 are detectable, the matrix has $2n$ LI columns for $s = \lambda$, where λ is an unstable eigenvalue of either A_1 or A_2. The composite system is therefore detectable.

Problem 2. a

$$\begin{pmatrix} \dot{\delta x} \\ \dot{\delta v} \\ \dot{\delta \psi} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta v \\ \delta \psi \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_A \\ E_G \end{pmatrix}$$

The characteristic polynomial of the matrix A above is : $\psi_A(s) = s^3 + \frac{g}{R}s$, with roots

$$\sigma(A) = \{0, \pm j\sqrt{\frac{g}{R}}\}$$

all of which are on the $j\omega$ axis. The size of the Jordan block corresponding to those roots is 1, so according to the criterion in LN 20 (page 6), the system is internally stable.

However, since we consider any possible output, the system has only marginal BIBO stability.

b.

$$\ddot{\delta v} = -g\dot{\delta \psi} = -\frac{g}{R}\delta v - gc$$

Hence

$$\delta v(t) = a\cos(\sqrt{\frac{g}{R}}t) + b\sin(\sqrt{\frac{g}{R}}t) - Rc$$

then

$$\delta x(t) = a\sin(\sqrt{\frac{g}{R}}t) - b\cos(\sqrt{\frac{g}{R}}t) - Rct + K$$

Problem 3 a. Assume that $\lambda_{\max}(A(t)) \leq -\epsilon$ for all t. This condition implies that the system is asymptotically stable. Defining $V(x) = x^T x$, we have:

$$\dot{V}(x) = x^T A^T(t)x + x^T A(t)x = 2x^T A(t)x \tag{5}$$

Since $A(t)$ is symmetric, we can form a dyadic decomposition:

$$A(t) = \sum_i \lambda_i(t)v_i(t)v_i(t)^T \tag{6}$$

where

$$\sum v_i(t)v_i(t)^T = I \tag{7}$$

From this we have:

$$\begin{aligned} &\dot{V}(x) = 2\textstyle\sum_i \lambda_i(t)x^T v_i(t)v_i(t)^T x \\ &\leq 2\lambda_{\max}(t)x(\textstyle\sum_i v_i(t)v_i^T(t))x \\ &= 2\lambda_{\max}(t)x^T x \\ &\leq -2\epsilon V(x) \end{aligned} \tag{8}$$

Integrating both sides of this inequality from 0 to t yields:

$$\|x(t)\|^2 = V(x) \leq V(x(0))e^{-2\epsilon t} \tag{9}$$

which shows that the system is exponentially (and hence asymptotically) stable.

(b). $\sigma(A) \subset \mathbf{C}_-^o \Leftrightarrow$ for all $Q = C^T C$ such that (A, C) is c.o., there exists a unique $P \succ 0$ such that

$$A^T P + PA = -Q \tag{10}$$

Proof: This is an extension of the proof for the case where $Q \succ 0$. Here, since $y^T Q y = \|Cy\|^2 \geq 0$ we have that Q is positive semidefinite but not definite.

($\Rightarrow$:) (This part is not asked by the problem, but it's interesting to know.) From the proof for the case where $Q \succ 0$, we can reuse the fact that the map

$\mathcal{L} : P \mapsto A^T P + PA$ is nonsingular since its eigenvalues $\{\lambda_i + \lambda_j\}$ are all nonzero for $\sigma(A) \subset \mathbf{C}_-^o$. We can also reuse the integral form for the solution P

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt \tag{11}$$

since when verifying this by substitution we did not use the fact that $Q \succ 0$. Now to show that $P \succ 0$. Using $Q = C^T C$, we have

$$x^T P x = \int_0^\infty \|Ce^{At}x\|^2 \tag{12}$$

We claim that for $x \neq 0$, the vector $Ce^{At}x \neq 0$ for all t. Suppose that for some $x \neq 0$, we have $Ce^{At}x \equiv 0$. Then the derivatives of this function of t will also be 0:

$$\begin{aligned} &Ce^{At}x = 0 \\ &CAe^{At}x = 0 \\ &\vdots \\ &CA^{n-1}e^{At}x = 0 \end{aligned} \tag{13}$$

which we can rewrite as

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} e^{At}x \equiv 0 \tag{14}$$

Since (C, A) is c.o., the observability matrix have full column rank. This implies that $e^{At}x \equiv 0$. But since e^{At} is guaranteed to be nonsingular, this implies that $x = 0$, a contradiction. Therefore the quantity $\int_0^\infty \|Ce^{At}x\|^2 > 0$ for all $x \neq 0$ and we have shown that $P \succ 0$.

($\Leftarrow$:) Observe that (C, A) is not c.o. iff there exists a nonzero vector v such that

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} v = 0 \tag{15}$$

or

$$\begin{aligned} \lambda v &= Av \\ Cv &= 0 \end{aligned} \tag{16}$$

Therefore (C, A) is c.o. iff none of the eigenvectors of A are in the nullspace of C. Now suppose that for (A, C) c.o. we have $A^*P + PA = -C^*C$. Let v be an eigenvector of A, writing $Av = \lambda v$ and $v^*A^* = \bar{\lambda}v^*$. Using this we obtain:

$$v^*A^*Pv + v^*PAv = -v^*C^*Cv \tag{17}$$

or

$$2\text{Re}(\lambda)v^*Pv = -\|Cv\|^2 \tag{18}$$

Since (C, A) is c.o., as we showed above, $\|Cv\|^2 > 0$. This means that the right hand side is strictly negative, while $v^* P v > 0$ since P is positive definite. Therefore we must have that $\text{Re}(\lambda) < 0$.

Problem 4 (a). Recall that $\mathcal{L} : P \mapsto A^T P + PA$ has e-values $\{\lambda_i + \lambda_j\}, i, j = 1, 2, \ldots, n$ where λ_i, λ_j are e-values of A (Note that if A has repeated e-values, $\mathcal{L}$ also has repeated e-value; if A doesn't have n L.I. e-vectors, $\mathcal{L}$ does not has n^2 L.I. e-vectors).

$\because \lambda_i + \lambda_j \neq 0, \forall i, j \therefore \mathcal{L}$ is non-singular. (P can be written as a n^2 by 1 vector, and $\mathcal{L}$ can be represented as a n^2 by n^2 matrix)

Therefore, the equation has a unique solution.

Suppose the unique solution P is not symmetric, i.e., $P^T \neq P$ but $A^T P + PA = Q$. Then $(A^T P + PA)^T = A^T P^T + PA = Q^T = Q$, which means that P^T is also a solution. Contradiction. Therefore P is symmetric.

(b). If $P = \int_0^\infty e^{A^T t} Q e^{At} dt$, then

$$
\begin{aligned}
A^T P + PA &= \int_0^\infty A^T e^{A^T t} Q e^{At} dt + \int_0^\infty e^{A^T t} Q e^{At} A dt \\
&= \int_0^\infty (\frac{d}{dt} e^{A^T t}) Q e^{At} dt + \int_0^\infty e^{A^T t} Q (\frac{d}{dt} e^{At}) dt \\
&= \int_0^\infty \frac{d}{dt} (e^{A^T t} Q e^{At}) dt \\
&= e^{A^T t} Q e^{At} |_0^\infty \\
&= -Q
\end{aligned}
$$

The last equality follows from the fact that $\sigma(A) \subset \mathcal{C}_-^o$.