1 Lipschitz Continuity

Definition 1. $f$ is globally Lipschitz continuous (LC) if there exists $L$ such that

$$\|f(x) - f(y)\| \leq L \|x - y\|,$$

for all $x, y \in \mathbb{R}^n$.

$f$ is locally Lipschitz continuous in $U \subset \mathbb{R}^n$, if for every $x, y \in U$, the Lipschitz property above is satisfied.

Consider. Which norm should we use to check the Lipschitz condition?

Problem 1. (Local or global Lipschitz condition.) Consider the following system of differential equations:

$$\begin{align*}
\dot{x}_1 &= x_2^2 + x_2^2 \\
\dot{x}_2 &= x_1^2 - x_2^2
\end{align*}$$

Prove that this system is locally Lipschitz, but not globally Lipschitz.

2 Fundamental Theorem

Theorem 2 (Fundamental Theorem of Differential Equations). Consider the following ordinary differential equation (ODE):

$$\begin{align*}
\dot{x} &= f(x, t), \\
x(t_0) &= x_0,
\end{align*}$$

with the vector field $f : \mathbb{R}^{n} \times \mathbb{R} \rightarrow \mathbb{R}^{n}$. If $f$ is

- piecewise continuous in $t$
- Lipschitz continuous in $x$,

then the ODE admits a unique solution, which is differentiable almost everywhere except at points where $f$ is discontinuous with respect to $t$. 
Problem 2. (Linear systems) Consider the following linear system:

\[
\dot{x} = A(t)x(t) + B(t)u(t),
\]

\[x(t_0) = x_0.\]

Provide a sufficient condition for the linear system to have a unique solution.

3 Bellman-Gronwall lemma

Theorem 3 (Bellman-Gronwall Inequality). Let \( u(\cdot) \) be a nonnegative, piecewise continuous function on \([0, T]\) which satisfies

\[
\begin{align*}
u(t) & \leq C_1 + \int_{t_0}^{t} k(\tau) u(\tau) d\tau, \\
\end{align*}
\]

for some constant \( C_1 \geq 0 \) and a nonnegative integrable function \( k \). Then

\[
\begin{align*}
\begin{align*}
u(t) & \leq C_1 \exp \left( \int_{t_0}^{t} k(\tau) d\tau \right), \\
\end{align*}
\end{align*}
\]

for \( 0 \leq t_0 < t \leq T \).

Problem 3. (Variation on linear systems) Consider the following linear system:

\[
\begin{align*}
\dot{x} & = Ax(t), \quad t \in (0, T] \\
x(0) & = x_0,
\end{align*}
\]

where the matrix \( A \) is in \( \mathbb{R}^{n \times n} \). Now we consider the variation \( x_0 + \tilde{x}_0 \) of the initial value, and the corresponding linear system:

\[
\begin{align*}
\dot{\tilde{x}} & = A\tilde{x}(t), \quad t \in (0, T] \\
\tilde{x}(0) & = x_0 + \tilde{x}_0.
\end{align*}
\]

Let \( \tilde{x} := \tilde{x} - x \) be the variation on the state. Then \( \tilde{x} \) solves the following linear system:

\[
\begin{align*}
\dot{\tilde{x}} & = A\tilde{x}(t), \quad t \in (0, T] \\
\tilde{x}(0) & = \tilde{x}_0.
\end{align*}
\]

Show that \( \|\tilde{x}(t)\| \to 0 \) as \( \|\tilde{x}_0\| \to 0 \) for any \( t \in [0, T] \) using Bellman-Gronwall lemma.
Problem 4. (Differential version) Let $x(t)$ be a nonnegative, continuously differentiable function on $[0,T]$, which satisfies
\[ \dot{x}(t) \leq A(t)x(t) + B(t) \]
for all $t \in [0,T]$, where $A$ and $B$ are nonnegative integrable functions on $[0,T]$. Show that
\[ x(t) \leq \exp \left( \int_0^t A(\tau) d\tau \right) \left[ x(0) + \int_0^t B(\tau) d\tau \right] \]
for all $t \in [0,T]$.

4 Dynamical Systems

$(\mathcal{U}, \mathcal{Y}, \Sigma, s, r)$: (input, state, output, state transition function, output read-out map).

- **Input**: $\mathcal{U} \subset \{ u : [0, \infty) \rightarrow U \mid U \text{ vector space (typically } \mathbb{R}^n) \}$ (Note that $\mathcal{U}$ is a function space.)
- **Output**: $\mathcal{Y} \subset \{ y : [0, \infty) \rightarrow Y \mid Y \text{ vector space (typically } \mathbb{R}^n) \}$ (Note that $\mathcal{Y}$ is a function space.)
- **State Space**: $\Sigma$, a vector space (typically $\mathbb{R}^n$)
- **State transition function**: $s : \mathbb{R} \times \mathbb{R} \times \Sigma \times \mathcal{U} \rightarrow \Sigma$ with $s(t, t_0, x_0, u[t_0, t]) = x(t)$
- **Output read-out map**: $r : \mathbb{R} \times \Sigma \times \mathcal{U} \rightarrow \mathcal{Y}$ with $r(t, x(t), u(t)) = y(t)$
- **Response function**: composition of $s$ and $r$: $\rho : \mathbb{R} \times \mathbb{R} \times \Sigma \times \mathcal{U} \rightarrow \mathcal{Y}$ with $\rho(t, t_0, x_0, u[t_0, t]) = y(t)$

Problem 5. Suppose that the dynamical system
\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t) \\
x(t_0) &= x_0
\end{align*}
\]
(1)

admits the unique solution
\[ x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau, \]
for $t \in [t_0, \infty)$. Identify the state transition function, the output read-out map and the response function.
Two axioms

1. State transition axiom: given $u_1, u_2 \in \mathcal{U}$ with $u_1(t) = u_2(t)$ for $t \in [t_1, t_2]$, we have

$$s(t_2, t_1, x_0, u_1[t_1, t_2]) = s(t_2, t_1, x_0, u_2[t_1, t_2])$$

2. Semi-group axiom: $\forall t_0 \leq t_1 \leq t_2$, $\forall x_0 \in \Sigma$, $\forall u \in \mathcal{U}$,

$$s(t_2, t_0, x_0, u[t_0, t_2]) = s(t_2, t_1, s(t_1, t_0, x_0, u[t_0, t_1]), u[t_1, t_2])$$

**Problem 6.** Show that the dynamical system (1) satisfies the two axioms above.