1 Linear Quadratic Regulator (LQR)

Consider the following discrete linear time-invariant dynamical system:

\[ x_{t+1} = Ax_t + Bu_t, \quad t \in \{0, 1, \ldots, N\} \]
\[ x_0 = x^{init}, \]

and the cost function:

\[ J(U, x_0) = \sum_{\tau=0}^{N-1} (x^T\tau Q x_\tau + u^T\tau Ru_\tau) + x^T_N Q_f x_N, \]

where \( Q, R, Q_f \geq 0 \) are positive semidefinite matrices and \( U := (u_0, u_1, \ldots, u_{N-1}) \). We also define the cost to go from state \( z \) at time \( t \) as:

\[ J^*_t(z) = \min_{u_t, \ldots, u_{N-1}} \sum_{\tau=t}^{N-1} (x^T\tau Q x_\tau + u^T\tau Ru_\tau) + x^T_N Q_f x_N, \]

where \( x_t = z \).

**Theorem 1.** The optimal cost-to-go and the optimal control at time \( t \) are given by:

\[ J^*_t(z) = z^T P_t z \]
\[ u^*_t = -K_t z, \]

where \( t \in \{0, 1, \ldots, N-1\} \) and

\[ P_t = Q + K^T_t R K_t + (A + B K_t)^T P_{t+1} (A + B K_t), \quad P_N = Q_f \]
\[ K_t = (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A. \]

**Remark 1.** LQR algorithm: To minimize the cost function in (2) subject to system dynamics in (1), we apply the control sequence \( \{-K_0 x^{init}, -K_1 x_1, \ldots, -K_{N-1} * x_{N-1}\} \), where \( x_i \) is obtained using \( x_{i-1} \) and \( u_{i-1} := -K_{i-1} * x_{i-1} \) through equation (1).

**Problem 1.** Let the system matrices in (1) are given by:

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]

and the cost matrices are given by \( Q = Q_f = I \) and \( R = \rho R I \). Let \( x_0 = (1, 0) \) and \( N = 20 \). Explain how states change under the optimal control feedback when (i) \( \rho_R = 1 \) and (ii) \( \rho_R = 10 \).
2 Extensions of LQR

2.1 Time-invariant State-affine Systems

The general dynamics of a time-invariant state-affine system are given by:

\[ x_{t+1} = Ax_t + Bu_t + c, \quad t \in \{0, 1, \ldots, N\} \tag{6} \]
\[ x_0 = x^{\text{init}} \]

**Problem 2.** Derive the optimal LQR control policy that minimizes (2) subject to state-affine dynamics in (6). Also derive the optimal cost-to-go.

2.2 Trajectory Following for LTI Systems

Suppose that now we want to derive a control policy that minimizes the deviation of our system’s trajectory from a reference trajectory \((x^*_i, u^*_i), i \in \{0, 1, \ldots, N\}\) (in other words, we want to follow the reference trajectory as closely as possible). The cost function in this case is given by:

\[ J(U, x_0) = \sum_{\tau=0}^{N-1} ((x_\tau - x^*_\tau)^T Q (x_\tau - x^*_\tau) + (u_\tau - u^*_\tau)^T R (u_\tau - u^*_\tau)) + (x_N - x^*_N)^T Q_f (x_N - x^*_N), \tag{7} \]

**Problem 3.** Derive the optimal LQR control policy and cost-to-go function that minimize (7) subject to the system dynamics in (1).

**Consider.** What would happen if the reference trajectory \((x^*, u^*)\) is not dynamically feasible?

2.3 Linear Time-variant (LTV) Systems

The general dynamics of a LTV system are given by:

\[ x_{t+1} = A_t x_t + B_t u_t, \quad t \in \{0, 1, \ldots, N\} \tag{8} \]
\[ x_0 = x^{\text{init}} \]

**Problem 4.** Derive the optimal LQR control policy and the cost-to-go function for LTV systems.
Remark 2. We can similarly derive the optimal LQR policy for time-variant state-affine systems or for trajectory following.

2.4 Trajectory Following for Non-linear Systems

We will use slides for this subsection.

2.5 Group work

Problem 5. Derive the optimal LQR control policy and the cost-to-go function for the reference trajectory problem for LTI systems when the reference trajectory \((x^*, u^*)\) is not dynamically feasible.

Problem 6. Suppose that now our cost function in (2) also includes a penalty on change in the control. So the cost function is now given by:

\[
J(U, x_0) = \sum_{\tau=0}^{N-1} (x_\tau^T Q x_\tau + u_\tau^T R u_\tau) + x_N^T Q_f x_N + \sum_{\tau=0}^{N-2} \Delta u_\tau^T S \Delta u_\tau, \tag{9}
\]

where \(\Delta u_\tau \equiv u_{\tau+1} - u_\tau\). Derive the optimal LQR control policy and the cost-to-go function for this new cost function.

Problem 7. Discuss how you will solve the above two problems for LTV systems.

2.6 Advance Problems

Problem 8. Consider the following optimal control problem, which considers a linear system with additive noise and quadratic cost:

\[
\min_{x, u} \sum_{\tau=0}^{N-1} E \left[ x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right] + E \left[ x_N^T Q_f x_N \right]
\]

subject to

\[
x_{t+1} = Ax_t + Bu_t + w_t, \ t \in \{0, 1, \ldots, N\} \tag{10}
\]

\[
x_0 = x^{init},
\]
with $w_t$ independent random vectors with $E[w_t] = 0$, and $E[w_t w_t^T] = \Sigma_w$. Find an LQR-like sequence of matrix updates that computes the optimal cost-to-go at all times and the optimal feedback controller at all times. Describe the expected cost incurred in excess of the expected cost in the case when there is no noise.

*Remark 3.* When $w_t$ in the problem above are drawn from the Gaussian distribution, then it is called a Linear Quadratic Gaussian (LQG) controller.