Your answers must be supported by analysis, proof, or counterexample. You may quote results that we derived in class in your solutions, however if the problem asks you to prove something that was done in class, I expect a step by step proof.

There are 9 questions: Please make sure your exam paper has all 9 questions.

Approximate points for each question are indicated. The exam is out of 48 points total.

You are allowed to use 1 8.5 x 11 crib sheet (both sides).
Problem 1: Internal exponential stability implies BIBO stability (6 points).
Consider the LTI system \( \dot{x} = Ax + Bu, \ y = Cx \).

(a) Prove that if this system is internally exponentially stable, then it is BIBO stable.
(b) Does the same result hold if the system is only internally stable? Prove or give a counterexample.
Problem 2: Controllability (5 points).

For an LTI system $R = (A, B, C, D)$ with $A \in \mathbb{R}^{n \times n}$, directly prove the standard controllability result that $\text{rank}[sI - A|B] = n, \forall s \in \mathbb{C}$ implies that $\text{rank}[B \ AB \ \cdots \ A^{n-1}B] = n$. 


Problem 3: Characteristic and minimal polynomials (3 points).

Let the characteristic polynomial of $A$ be

$$\chi_A(s) = (s - \lambda)^5$$

and its minimal polynomial be $\Psi_A(s) = (s - \lambda)^3$. True or False: there exists a nonsingular matrix $P$ such that

$$PAP^{-1} = \begin{bmatrix}
\lambda & 1 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda
\end{bmatrix}$$

(2)

Explain your answer.
Problem 4: Stability (4 points).

You are given a SISO transfer function

\[ \frac{(s + 1)(s + 3)}{s^2(s + 2)^2(s + 4)} \]

and are told that it has a minimal (controllable and observable) realization $A, b, c$ with $A \in \mathbb{R}^{5 \times 5}, b \in \mathbb{R}^5, c \in \mathbb{R}^{1 \times 5}$. Is the system without input:

\[ \dot{x} = Ax \]

Problem 5: Stabilizability and Detectability (4 points).

Consider the LTI system \( \dot{x} = Ax + Bu, \ y = Cx \) where

\[
A = \begin{bmatrix}
-1 & 0 \\
0 & 3
\end{bmatrix}, \ B = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \ C = [0 \ 1]
\]

Is the system stabilizable? Detectable? Prove your answers.
Problem 6: Properties of interconnected subsystems (8 points).

Consider the interconnected systems (i) and (ii) shown in Figure 1. For each of these systems,

(a) (4 points) Determine the internal stability of the resulting interconnection.

(b) (4 points) Is the resulting interconnection controllable? Observable? Explain.
Problem 7: Controllability and observability (6 points).

Consider the LTI system $\dot{x} = Ax + Bu$.

(a) (2 points) Suppose that this open loop system is controllable. Show that the closed loop system resulting from state feedback $u = Fx + v$ is controllable (from new input $v$).

(b) (4 points) Now, in addition, assume that $y = Cx$ and that the open loop system is observable. Suppose that $\psi(y)$ is a known, possibly nonlinear, function of $y$. Show, by designing an appropriate observer and analyzing the convergence of the state estimate error, that the system resulting from output injection: $\dot{x} = Ax + \psi(y) + Bu$, $y = Cx$, is observable. (Hint: consider the block diagram of the observer)
Problem 8: Reachable states (5 points).

Given the system
\[
\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u; \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\] (3)

True or false: there exists \( u[0, T] \) such that \( x(T) = [1, 1, 0]^T \).

If true, give a sketch of the proof, if false, explain.
Problem 9: Combined observer and controller: closed loop properties (7 points). Consider LTI systems described by the equations $\dot{x} = Ax + Bu, \; y = Cx + Du$ where $A \in \mathbb{R}^{n \times n}, \; B \in \mathbb{R}^{n \times n_1}, \; C \in \mathbb{R}^{n_2 \times n},$ and $D \in \mathbb{R}^{n_2 \times n_1}$. Suppose that you had designed a full order observer (using observer gain matrix $T$) forming an estimate $\hat{x}(t)$ of the state, and then used that estimate in state feedback with the control law $u = F\hat{x} + r$.

(a) Determine the closed loop transfer function between $r$ and $y$. What is remarkable about it?

(b) Now, with $e(t) = x(t) - \hat{x}(t)$, determine the closed loop transfer function between $e(0)$ and $y$ and discuss the effect of $e(0)$ on $y$. 