Problem 1.

- \( f \) is a function: it has well defined domain and codomain, and \( \forall x \in \mathbb{R}^3, \exists! f(x) \in \mathbb{R}^3 \).
- \( f \) is not injective: for \( e_1 = [1 \ 0 \ 0]^T \) and \( e_2 = [0 \ 1 \ 0]^T \), \( f(e_1) = f(e_2) \) but \( e_1 \neq e_2 \).
- \( f \) is not surjective: \( \nexists x \) such that the third component of \( f(x) \neq 0 \), so the range of \( f \) is not equal to the codomain \( \mathbb{R}^3 \).

Problem 2.

(a) Define addition as XOR (or addition modulo 2) and multiplication as AND (or standard multiplication). Check that these satisfy the axioms of a field.

(b) Assume the standard definitions of addition and multiplication for matrix arithmetic. The \( n \times n \) identity matrix \( I \) is nonsingular. So \( I \in \text{GL}_n \). However, \( I + (-I) = 0 \) is singular. Therefore, it is not a field. (Also: matrix multiplication is not commutative).

Problem 3.

(a) The key point here is that you cannot just assume that associativity, commutativity and distributivity holds in \( \mathbb{R}^n \) – this is what you are supposed to show! The clean way to argue this is to realize that all operations are performed element-wise, and that the desired properties follow from the properties of the field \( \mathbb{R} \) (all elements of a vector \( x \in \mathbb{R}^n \) are themselves members of \( \mathbb{R} \), so properties of \( \mathbb{R} \) can be applied element-wise).

(b) We can write a vector in the set as \( x(s) = a_0 + a_1 s + \ldots + a_{k-1} s^{k-1} + a_k s^k \), where \( a_i \in \mathbb{R} \) for \( i = 0, 1, \ldots, k \). For vector addition, associativity and commutativity can be shown by looking at the operations component-wise. The additive identity \( \theta \) is the zero polynomial \( (a_0 = a_1 = \ldots = a_k = 0) \) and the additive inverse of a vector just has each coefficient negated. The axiom of scalar multiplication and the distributive laws can be shown by considering operations component-wise and noting that the coefficients \( a_i \) also belong to the field \( \mathbb{R} \).

A natural basis is \( B := \{1, s, s^2, \ldots, s^k\} \). It spans the space (we can write a general \( x(s) \) as linear combinations of the basis elements) and they are linearly independent since \( a_0 + a_1 s + \ldots + a_k s^k = \theta \) only if \( a_0 = a_1 = \ldots = a_k = 0 \). The dimension of the vector space is thus the cardinality of \( B \), which is \( k+1 \).
Problem 4.
In order for the plane \( W \) to be a subspace of the vector space \( \mathbb{R}^3 \), we require that \( W \subseteq \mathbb{R}^3 \) and that \( W \) is itself a vector space. A plane in \( \mathbb{R}^3 \) is a two dimensional surface, described by \( W = \{ x \mid c^T x = b \} \) for some \( c \in \mathbb{R}^3, b \in \mathbb{R} \). By our statement that this is a plane in \( \mathbb{R}^3 \), we have restricted the elements \( x \) to \( \mathbb{R}^3 \), so \( W \subseteq \mathbb{R}^3 \). Because \( W \) is a subset of a vector space, it inherits properties of that space (associativity and commutativity of vector addition, as well as the properties of scalar multiplication (other than closure) and distributivity). What remains to be shown is that \( W \) is closed under vector addition and scalar multiplication (which also implies that \( W \) contains the additive identity \( \theta \) of \( \mathbb{R}^3 \) and that all elements \( x \) have an additive inverse \(-x\)).

If and only if the plane passes through the origin (\( \theta \in W \)), then we have that the parameter \( b \) (from the definition of the plane \( W \)) is equal to zero. For any vectors \( x_1, x_2 \in W \), and scalars \( \alpha, \beta \in \mathbb{R} \), we have that

\[
c^T (\alpha x_1 + \alpha x_2) = \alpha c^T x_1 + \alpha c^T x_2 = \alpha 0 + \alpha 0 = 0
\]

which implies that \((\alpha x_1 + \alpha x_2) \in W\). Therefore, \( W \) is closed under vector addition and scalar multiplication and is thus itself a vector space.

If the plane does not contain the origin (\( \theta \notin W \)), then it is the case that for all \( x \in W \), \( 0x = \theta \notin W \), so the set is not closed under scalar multiplication and is not a vector (sub)space.

Problem 5.
(a) Yes, it is a subspace. First, \( U_1 + \ldots + U_m \) is a subset since its elements are sums of vectors in subspaces (hence also subsets) of \( V \) and since \( V \) is a vector space, those sums are also in \( V \). Let us choose \( u_i, \tilde{u}_i \) in \( U_i \) for \( i = 1, \ldots, m \). Then, we have

\[
\alpha \sum_{i=1}^{m} u_i + \beta \sum_{i=1}^{m} \tilde{u}_i = \sum_{i=1}^{m} (\alpha u_i + \beta \tilde{u}_i) \in U_1 + \cdots + U_m,
\]

since \( \alpha u_i + \beta \tilde{u}_i \in U_i \). Hence \( U_1 + \ldots + U_m \) is also closed.

(b) Counterexample: \( U_1 = \{ \theta \} \) and \( U_2 = W \neq U_1 \). Then \( U_1 + W = W = U_2 + W \). Also consider the case where \( W = V \). It is also easy to find non-trivial counterexamples.

Problem 6. \( \{ f_k \}_{k=0}^{\infty} \) is clearly a subset of the set of all sequences of real numbers. Let \( f, g \in \{ f_k \}_{k=0}^{\infty} \) and \( \alpha, \beta \in \mathbb{R} \). Then

\[
\{ \alpha f + \beta g \}_{k} = \alpha f_k + \beta g_k = \alpha (f_{k-1} + f_{k-2}) + \beta (g_{k-1} + g_{k-2}) = (\alpha f_{k-1} + \beta g_{k-1}) + (\alpha f_{k-2} + \beta g_{k-2}) = \{ \alpha f + \beta g \}_{k-1} + \{ \alpha f + \beta g \}_{k-2}
\]

This shows that \( \{ f_k \}_{k=0}^{\infty} \) closed, so \( \{ f_k \}_{k=0}^{\infty} \) is a subspace.

Problem 7. The set \( \{ v_1, v_2 \} \) is linearly dependent over the complex valued rational functions; take \( \alpha = \frac{s+3}{s+2} \). Then \( v_1 = \alpha v_2 \). On the other hand, \( \{ v_1, v_2 \} \) is linearly independent over \( \mathbb{R} \); fix \( \alpha, \beta \in \mathbb{F} \). Then \( \alpha v_1 + \beta v_2 = 0 \) implies \( \alpha(s + 3) + \beta(s + 2) = 0 \) for all \( s \in \mathbb{C} \), further implying that \( \alpha = \beta = 0 \).
Problem 8. Solution 1

Not linearly independent: $A^2 - 3A + 2I = 0$. cf. Cayley-Hamilton Theorem (later in class).

Solution 2

Set $\{I, A, A^2\}$ is linearly independent if and only if

$$\alpha I + \beta A + \gamma A^2 = 0; \alpha, \beta, \gamma \in \mathbb{R} \implies \alpha = \beta = \gamma = 0,$$

where $0$ denotes $2 \times 2$ zero matrix. However, the equation $\alpha I + \beta A + \gamma A^2 = 0$ holds for any $\gamma \in \mathbb{R}$, $\alpha = 2\gamma, \beta = -3\gamma$. Choosing any $\gamma \neq 0$ and corresponding $\alpha, \beta$ thus implies that the set $\{I, A, A^2\}$ is linearly dependent.

Problem 9. The first set is linearly independent: The matrix obtained by stacking the vectors has non-zero determinant. Alternatively, this can be seen by determining the rank after performing elementary row and column operations. The second set of vectors is linearly dependent: $[4 \quad 5 \quad 1]^T = 2[1 \quad 2 \quad -1]^T + [2 \quad 1 \quad 3]^T$.

Problem 10. $B := \{b_1, b_2, b_3\} = \{[1, 1/3, 0, 0, 0]^T, [0, 0, 1, 0, 1/7]^T, [0, 0, 0, 0, 1]^T\}$ is a basis. The first two vectors $b_1$ and $b_2$ enforce the restrictions on $U$, and $b_3$ provides the remaining degree of freedom. We can check that $B$ is linearly independent and that span($B$) = $U$.

Problem 11. Fix $\{\alpha_i\}_{i=1}^n \in \mathbb{F}$. Note that

$$\alpha_1(v_1 - v_2) + \alpha_2(v_2 - v_3) + \ldots + \alpha_n v_n = 0 \iff \alpha_1 v_1 + (\alpha_2 - \alpha_1) v_2 + \ldots + (\alpha_n - \alpha_{n-1}) v_n = 0$$

Now, since $\{v_1, \ldots, v_n\}$ is linearly independent, the above inequality implies that $\alpha_1 = \alpha_2 - \alpha_1 = \ldots = \alpha_n - \alpha_{n-1} = 0$. Therefore $\alpha_1 = 0$, thus $\alpha_2 = 0$, etc. Hence $\alpha_i = 0$ for all $i$, which implies that $\{v_1 - v_2, \ldots, v_{n-1} - v_n, v_n\}$ is linearly independent.

Problem 12.

(a) Linear: $A(\alpha u(t) + \beta v(t)) = \alpha u(-t) + \beta v(-t) = \alpha A(u(t)) + \beta A(v(t))$.

(b) Linear:

$$A(\alpha u(t) + \beta v(t)) = \int_0^t e^{-\sigma}[\alpha u(t - \sigma) + \beta v(t - \sigma)]d\sigma$$

$$= \alpha \int_0^t e^{-\sigma}u(t - \sigma)d\sigma + \beta \int_0^t e^{-\sigma}v(t - \sigma)d\sigma = \alpha A(u(t)) + \beta A(v(t)).$$

(c) Linear:

$$A(\alpha a_1 s^2 + b_1 s + c_1) + \beta (a_2 s^2 + b_2 s + c_2)) = \int_0^s (ab_1 + \beta b_2)t + (\alpha a_1 + \beta a_2)dt$$

$$= \alpha \int_0^s b_1 t + a_1 dt + \beta \int_0^s b_2 t + a_2 dt$$

$$= \alpha A(a_1 s^2 + b_1 s + c_1) + \beta A(a_2 s^2 + b_2 s + c_2).$$
Problem 13. Suppose that \(\dim \mathcal{N}(A) = k\). If \(k = n\), then \(\mathcal{R}(A) = \{0\}\), which has dimension zero, and the expression holds. Let \(\{u_1, \ldots, u_k\}\) be a basis for \(\mathcal{N}(A)\). We may extend \(\{u_1, \ldots, u_k\}\) to a basis \(\{u_1, \ldots, u_n\}\) for \(U\). We claim that \(\{A(u_{k+1}), \ldots, A(u_n)\}\) is a basis for \(\mathcal{R}(A)\). Observe first that \(\mathcal{R}(A) = \text{span} \{A(u_1), \ldots, A(u_n)\} = \text{span} \{A(u_{k+1}), \ldots, A(u_n)\}\). Moreover, suppose that \(\sum_{i=k+1}^n b_i A(u_i) = 0\) for some \(b_{k+1}, \ldots, b_n \in \mathbb{F}\). Then \(A(\sum_{i=k+1}^n b_i u_i) = 0\). Hence, \(\sum_{i=k+1}^n b_i u_i \in \mathcal{N}(A)\). Therefore, there exist \(c_1, \ldots, c_k\) such that \(\sum_{i=k+1}^n b_i u_i = \sum_{i=1}^k c_i u_i\). However, since \(\{u_1, \ldots, u_n\}\) is linearly independent, we must have \(b_{k+1} = \ldots = b_n = 0\). Thus we conclude that the set \(\{A(u_{k+1}), \ldots, A(u_n)\}\) is linearly independent. Thus we have shown that \(\{A(u_{k+1}), \ldots, A(u_n)\}\) is a basis for \(\mathcal{R}(A)\). Hence, \(\dim \mathcal{R}(A) = n - k = \dim(U) - \dim \mathcal{N}(A)\), which implies that \(\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n\).