Problem 1. To find the eigenvalues of the matrix $A$, we need to find the roots of its characteristic equation

\[ \hat{\chi}_A(s) = \det(sI - A) = \det \begin{bmatrix} s + 1 & -3 & 1 \\ 3 & s - 5 & 1 \\ 3 & -3 & s - 1 \end{bmatrix} \]

\[ = (s + 1)((s - 5)(s - 1) + 3) + 3(3(s - 1) - 3) - (-9 - 3(s - 5)) \]

\[ = (s - 1)(s - 2)^2 \]

Once we have the eigenvalues ($\lambda_1 = 1$, $\lambda_2 = 2$), we can find the eigenvectors by solving the equation $(A - \lambda I)x = \theta_3$.

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2a + 3b - c \\ -3a + 4b - c \\ -3a + 3b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies a = b = c
\]

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3a + 3b - c \\ -3a + 3b - c \\ -3a + 3b - c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies [b = a, c = 0] \text{ OR } [b = 0, c = -3a]
\]

Therefore, we select the eigenvectors

\[ e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \implies T^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix} \]

The diagonalized form of $A$ is given by

\[ \Lambda = TAT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]

To compute $\cos(e^A)$, we note that $\cos(e^A)$ is an analytic function of $A$ and can thus be locally expressed as a convergent power series. For a diagonal matrix, raising that matrix to some power is equivalent to raising its diagonal entries to that same power. Therefore, applying an analytic function to a diagonal matrix is the same as applying that function to each diagonal entry. Furthermore, if we raise the matrix $(TAT^{-1})$ to some power $n$, we obtain $(TAT^{-1})(TAT^{-1})\cdots(TAT^{-1}) = T A^n T^{-1}$. We can thus find $\cos(e^A)$ by
computing $T^{-1}\cos(e^{TAT^{-1}})T$.

$$
\cos(e^{TAT^{-1}}) = \begin{bmatrix}
\cos(e^1) & 0 & 0 \\
0 & \cos(e^2) & 0 \\
0 & 0 & \cos(e^2)
\end{bmatrix} = \begin{bmatrix}
-0.912 & 0 & 0 \\
0 & 0.448 & 0 \\
0 & 0 & 0.448
\end{bmatrix}
$$

$$
\cos(e^A) = T^{-1}\cos(e^{TAT^{-1}})T = \begin{bmatrix}
-3.632 & 4.08 & -1.36 \\
-4.08 & 4.529 & -1.36 \\
-4.08 & 4.08 & -0.912
\end{bmatrix}
$$

**Problem 2.** We know that the determinant of a matrix is equal to the product of its eigenvalues. We also know that $e^A$ is given by a convergent power series.

$$
e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}
$$

If we post-multiply this expression by $v$, an eigenvector of $A$ associated to eigenvalue $\lambda$, we obtain

$$
e^A v = \sum_{n=0}^{\infty} \frac{A^n v}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n v}{n!} = e^{\lambda} v
$$

Because (1) holds for all eigenvectors of $A$, and the eigenvectors of a polynomial of a matrix are the eigenvalues of that matrix, the $i$-th eigenvalue of $e^A$ will be $e^{\lambda_i}$.

Any product of terms of the form $e^{ci}$ will not be equal to zero (they can only approach zero as $\sum_i c_i$ approaches $-\infty$). Therefore, the determinant of $e^A$ (which will be the product of eigenvalues $e^{\lambda_i}$) will not be equal to zero. This result is independent of the eigenvalues and the determinant of $A$.

**Problem 3.** Note that

$$
\hat{\chi}_A(s) = \det(sI - \bar{A}) = \det(sPP^{-1} - PAP^{-1}) = \det(P(sI - A)P^{-1}) = \det P \cdot \det(sI - A) \cdot \det(P^{-1})
$$

since $\det(AB) = \det A \cdot \det B$. In particular, we have $1 = \det I = \det P \cdot \det(P^{-1})$ and therefore $\hat{\chi}_A(s) = \det(sI - A) = \hat{\chi}_A(s)$. Since $\bar{A}$ and $A$ have the same characteristic polynomial, they have the same eigenvalues.

**Problem 4.** The number of Jordan blocks equals the number of linearly independent eigenvectors, so we know that we will have four total blocks. Furthermore, we know from the characteristic polynomial that there will be three total dimensions related to $\lambda_2$, and we are given that the largest Jordan block associated to $\lambda_2$ is of size 3, so we can uniquely determine that there will be one Jordan block associated to $\lambda_2$ and that it will be of size 3. We now seek a combination of Jordan block sizes for $\lambda_1$ such that there are three blocks, the largest of which is size 2, and the total dimensionality of which is 5. The unique solution to
these requirements is to have two Jordan blocks of size 2 and one of size 1. Therefore, we can determine the Jordan matrix $J$ (up to a permutation of Jordan blocks) as

$$
J = \begin{bmatrix}
\lambda_1 & 1 & 0 & \lambda_1 \\
0 & \lambda_1 & 1 & 0 \\
\lambda_2 & 1 & 0 & \lambda_2 \\
0 & \lambda_2 & 1 & 0 \\
0 & 0 & \lambda_2
\end{bmatrix}
$$

where $J = TAT^{-1}$.

**Problem 5.** We know that there is a single eigenvalue $\lambda = 0$ with algebraic multiplicity 6, and that the size of the largest Jordan block is 3. We know that $\text{rank}(A) = \text{rank}(T^{-1}JT) = \text{rank}(J)$ since $T$ is full rank (apply Sylvester’s inequality). Then $J$ must have rank of at least 2, arising from the ones on the superdiagonal in the Jordan block of size 3 (the diagonal entries will all be zeros since zero is the only eigenvalue). If all the other Jordan blocks were size 1, then there would be no additional ones on the superdiagonal, so the lower bound on $\text{rank}(A)$ is 2. Now the highest number of ones on the superdiagonal that this matrix could have is 4, which would be the case if there were two Jordan blocks of size 3. So $\text{rank}(A) \leq 4$. Thus the bounds are $2 \leq \text{rank}(A) \leq 4$.

**Problem 6.**

1. Since this matrix is upper triangular (indeed, it is already in Jordan form) we can simply read off the eigenvalues from the diagonal: $\sigma(A) = \{-3, -4, 0\}$. Since there are 4 Jordan blocks, there are also 4 linearly independent eigenvectors (two associated to the eigenvalue $\lambda = 0$ and one associated to each $\lambda = -3$ and $\lambda = -4$). The number of generalized eigenvectors associated with a Jordan block will be the size of the block minus one. Therefore, we can see that $-3$ has two generalized eigenvectors and $-4$ has one generalized eigenvector. Keep in mind, however, that technically, regular eigenvectors are “generalized eigenvectors of rank 1.”

2. By the spectral mapping theorem, $\sigma(e^{At}) = \{e^{-3t}, e^{-4t}, 1\}$.

**Problem 7.** Let $v \in V := \text{span}\{b, Ab, A^2b, \ldots, A^{n-1}b\}$. Then $v = \alpha_0 b + \alpha_1 Ab + \cdots + \alpha_{n-1} A^{n-1}b$ for some set of coefficients $\{\alpha_i\}_{i=0}^{n-1} \in \mathbb{F}^n$. Thus

$$
Av = \alpha_0 Ab + \alpha_1 A^2b + \cdots + \alpha_{n-2} A^{n-1}b + \alpha_{n-1} A^n b
$$

From the Cayley-Hamilton Theorem, we have $A^n = \beta_0 I + \beta_1 A + \cdots + \beta_n A^{n-1}$. Therefore

$$
Av = (\alpha_{n-1} \beta_0) b + (\alpha_0 + \alpha_{n-1} \beta_1) Ab + (\alpha_1 + \alpha_{n-1} \beta_2) A^2b + \cdots + (\alpha_{n-2} + \alpha_{n-1} \beta_{n-1}) A^{n-1}b
$$

$$
= \gamma_0 b + \gamma_1 Ab + \cdots + \gamma_{n-1} A^{n-1}b
$$

and so $Av \in V$. Hence, $V$ is an $A$-invariant subspace.