Problem 1. (a) Choosing $x_1 = T_C$ and $x_2 = T_H$ we have $\dot{x} = Ax + Bu$ and $y = x$, where

$$A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

after plugging in numerical values for the constants. We find the eigenvalues as $\lambda_1 = -\frac{1}{10}$ and $\lambda_2 = -\frac{1}{2}$, with associated eigenvectors $v_1 = [1 \ 1]^T$ and $v_2 = [1 \ -1]^T$. We have $A = T^{-1}JT$, where $T^{-1} = [v_1 \ v_2]$ and $J = \text{diag}(\lambda_1, \lambda_2)$.

Define $z = Tx$. Then

$$\dot{z} = Jz + TBu$$

$$y = T^{-1}z$$

where $TB = \frac{1}{20} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(b) We find that

$$y(t) = T^{-1}z(t) = T^{-1}e^{Jt}z_0 = T^{-1}e^{Jt}Tx_0 = \frac{1}{2} \begin{bmatrix} e^{-0.1t} + e^{-0.5t} & e^{-0.1t} - e^{-0.5t} \\ e^{-0.1t} - e^{-0.5t} & e^{-0.1t} + e^{-0.5t} \end{bmatrix} x_0$$

(c) The system has both eigenvalues in the open left half plane, and thus is internally exponentially stable. Therefore it is also BIBO stable (see Lecture Notes 15, page 7).

Problem 2. The transfer function has poles at $\pm j$. Hence, the poles are not in the open left half plane. So the system is not BIBO stable (see Lecture Notes 14, page 7).

Problem 3. Note that the solution to NL system is given by:

$$x(t) = \Phi(t, t_0)x_0 + \int_{\tau = t_0}^{t} \Phi(t, \tau)h(x, \tau)d\tau$$

This can be verified by taking the derivative of $x(t)$ and using the fact that $\Phi(t, t_0) = A(t)\Phi(t, t_0)$.

Taking norm on both sides of (1), we get:

$$\|x(t)\| \leq \|\Phi(t, t_0)\|\|x_0\| + \int_{\tau = t_0}^{t} \|\Phi(t, \tau)\|\|h(x, \tau)\|d\tau$$

$$\leq Me^{-\alpha(t-t_0)}\|x_0\| + \int_{\tau = t_0}^{t} Me^{-\alpha(t-\tau)}\|h(x, \tau)\|d\tau$$

$$\leq Me^{-\alpha(t-t_0)}\|x_0\| + \int_{\tau = t_0}^{t} \beta Me^{-\alpha(t-\tau)}\|x(\tau)\|d\tau.$$
Using Bellman-Gronwall inequality for \( u(t) := \|x(t)\|e^{\alpha(t-t_0)} \), we get:

\[
 u(t) \leq M\|x_0\|e^{\beta M(t-t_0)}.
\]  

(6)

Thus,

\[
 \|x(t)\| \leq M\|x_0\|e^{(\beta M-\alpha)(t-t_0)}.
\]  

(7)

If \( \beta M - \alpha < 0 \), then system state-norm is bounded above by a decaying exponential and hence system is exponentially stable.

**Problem 4.** 1. The eigenvalue of \( A \) is 0. Since each jordan block corresponding to 0 is of size 1, system is internally stable in this case, even though the eigenvalue is on the imaginary axis (see Lecture Notes 14, page 6).

2. Again, the eigenvalue of \( A \) is 0. Since the jordan block corresponding to 0 is of size 2, system is not internally stable in this case.

**Problem 5.** The matrix \( A \) has a Jordan block \( J_0 \) of size 3 corresponding to a zero eigenvalue. For this block we have

\[
e^{J_0 t} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}
\]

and so \( \|e^{J_0 t}\| \) is not bounded. In particular, any non-zero component of the initial condition in the space spanned by the two generalized eigenvectors will blow up and have a contribution to the state component in the subspace spanned by the eigenvector that goes linearly or quadratically with time. Hence, the system is not stable.

**Problem 6.** See pp. 12–13 of Lecture Notes 15.

**Problem 7.** (a) Defining \( z := [\delta x \ \delta v \ \delta \psi]^{\top} \) we have \( \dot{z} = Az + Bu \), where

\[
 A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} E_A \\ E_G \end{bmatrix}
\]

From this we compute \( \chi_A(s) = s(s^2 + g/R) \), which shows that the system is internally stable. W.l.o.g. we may consider a single output, so suppose \( C = [c_1 \ c_2 \ c_3] \). The transfer function matrix is \( H(s) = \)
\[ C(sI - A)^{-1}B \text{ is} \]

\[ H(s) = \begin{bmatrix}
  c_1 & c_2 & c_3 \\
\end{bmatrix}
\frac{1}{s(s^2 + \frac{g}{R})}
\begin{bmatrix}
  * & * & * \\
  s & s^2 + \frac{g}{R} & * \\
  -g & -sg & s^2 \\
\end{bmatrix}
\begin{bmatrix}
  0 & 0 \\
  1 & 0 \\
  0 & 1 \\
\end{bmatrix} \]

\[ = \frac{1}{s(s^2 + \frac{g}{R})}
\begin{bmatrix}
  c_1 & c_2 & c_3 \\
\end{bmatrix}
\begin{bmatrix}
  s & -g \\
  s^2 & -sg \\
  s & s^2 \\
\end{bmatrix} \]

\[ = \begin{bmatrix}
  \frac{c_1 + c_2 s + \frac{c_3}{s}}{s^2 + \frac{g}{R}} & , & * \\
\end{bmatrix} \]

We know that the system is BIBO stable if and only if \( H(s) \) has no poles outside the open left half plane. But it is clear from the first element \( H_1(s) = \frac{c_1 + c_2 s + \frac{c_3}{s}}{s^2 + \frac{g}{R}} \) that the poles at \( s = \pm j\sqrt{\frac{g}{R}} \) cannot be cancelled, so the only way for them to not show up in \( H_1(s) \) is if \( c_1 = c_2 = c_3 = 0 \). Thus the system is BIBO stable only for the trivial output \( y = 0 \).

(b) With \( C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top \), assuming zero initial condition, and \( U(s) = \mathcal{L}(u(t)) = \frac{E_G s}{s} \) we find

\[ \delta X(s) = H(s)U(s) = C(sI - A)^{-1}BU(s) = \frac{-g}{s(s^2 + \frac{g}{R})} \frac{E_G s}{s} = -gE_G \frac{1}{s^2(s^2 + \frac{g}{R})} \]

Partial fraction expansion yields

\[ \frac{g}{s^2(s^2 + \frac{g}{R})} = \frac{R}{s^2} - \frac{R}{s^2 + \frac{g}{R}} \]

and so, transforming back to the time domain, we have

\[ \delta x(t) = E_G R (-t + \sqrt{\frac{R}{g}} \sin \sqrt{\frac{g}{R}} t) \]

Hence the position error keeps growing (in absolute value), and oscillates around \( \delta \bar{x} = -E_G R t \) with frequency \( \sqrt{\frac{g}{R}} \).

**Problem 8.** We can rewrite this equation as

\[ A^T P - \lambda P + PA - \lambda P = -Q \]

\[ (A^T - \lambda I)P + P(A - \lambda I) = -Q \]

\[ (A - \lambda I)^T P + P(A - \lambda I) = -Q \]

Now, based on the information given in the problem, we can apply the Lyapunov Theorem to the matrix \( (A - \lambda I) \), indicating that its eigenvalues lie in the open left half-plane. We can see that the eigenvalues of \( (A - \lambda I) \) are equal to the eigenvalues of \( A \) minus \( \lambda \) (all vectors are eigenvectors of \( \lambda I \), and if \( Av = \lambda v \), then \( (A - \lambda I)v = (a - \lambda)v \)). Since the real part of all eigenvalues \( \mu \) of \( (A - \lambda I) \) are less than zero (\( \text{Re}(\mu) < 0 \)), we can conclude that the real part of all eigenvalues \( a \) of \( A \) are less than \( \lambda \) (\( \text{Re}(a) < \lambda \)).

Note: This form of the Lyapunov equation/Theorem applies to real-valued \( A \) matrices, which is generally the case for dynamic systems.