Problem 1: Functions. Consider \( f : \mathbb{R}^3 \to \mathbb{R}^3 \), defined as
\[
f(x) = Ax, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad x \in \mathbb{R}^3
\]

Problem 2: Fields.
(a) Define addition and multiplication on \( \{0, 1\} \) to form a field. Show that your result is a field.
(b) Is \( GL_n \), the set of all \( n \times n \) nonsingular matrices, a field? Justify your answer.

Problem 3: Vector Spaces.
(a) Show that \( (\mathbb{R}^n, \mathbb{R}) \), the set of all ordered \( n \)-tuples of elements from the field of real numbers \( \mathbb{R} \), is a vector space.
(b) Show that the set of all polynomials in \( s \) of degree \( k \) or less with real coefficients is a vector space over the field \( \mathbb{R} \). Find a basis. What is the dimension of the vector space?

Problem 4: Subspaces.
Is a plane in \( \mathbb{R}^3 \) a subspace of \( (\mathbb{R}^3, \mathbb{R}) \)?
Problem 8: Linear Independence. Let
\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \]
Is the set \{I, A, A^2\} linearly dependent or independent in \(\mathbb{R}^{2\times 2}\)?

Problem 9: Linear Independence. Which of the following sets are linearly independent in \(\mathbb{R}^3\)?
\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}
\]

Problem 10: Bases. Let \(U\) be the subspace of \(\mathbb{R}^5\) defined by
\[ U = \{[x_1, x_2, \ldots, x_5]^T \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\} \]
Find a basis for \(U\).

Problem 11: Bases. Prove that if \(\{v_1, v_2, \ldots v_n\}\) is linearly independent in \(V\), then so is the set \(\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}\).

Problem 12: Linearity. Are the following maps \(A\) linear?
(a) \(A(u(t)) = u(-t)\) for \(u(t)\) a scalar function of time
(b) How about \(y(t) = A(u(t)) = \int_0^t e^{-\sigma} u(t - \sigma) d\sigma\)?
(c) How about the map \(A : as^2 + bs + c \to \int_0^s (bt + a) dt\) from the space of polynomials with real coefficients to itself?

Problem 13: Rank-Nullity Theorem. Let \(A\) be a linear map from \(U\) to \(V\) with \(\dim U = n\) and \(\dim V = m\). Show that
\[ \dim R(A) + \dim N(A) = n \]