Problem 1: Discrete-time LQR.
Consider the following optimal control problem where we are interested in controlling the output instead of state:

\[
\min_{U} \sum_{\tau=0}^{N-1} (y^T\tau Q y^\tau + u^T\tau R u^\tau)
\]
subject to

\[
\begin{align*}
x_{t+1} &= Ax_t + Bu_t, \quad t \in \{0,1,\ldots,N-1\} \\
y_t &= Cx_t, \\
x_0 &= x^{\text{init}}.
\end{align*}
\]

Here, \(U\) is the sequence of control inputs as in lecture. Find an LQR-like sequence of matrix updates that computes the optimal cost-to-go at all times and the optimal feedback controller at all times.

Problem 2: Discrete-time LQR.
In the above problem, suppose the system matrices are given by:

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]

and the cost matrices are given by \(Q = Q_f = \rho_Q I\) and \(R = \rho_R I\). Let \(x_0 = (1,0)\) and \(N = 20\). Explain how the output, control and cost-to-go change under the optimal feedback when (i) \(\rho_Q = 1, \rho_R = 1\), (ii) \(\rho_Q = 10^3, \rho_R = 1\) and (iii) \(\rho_Q = 1, \rho_R = 10^3\). You can use MATLAB, python, or whatever you like to solve the problem.

Problem 4: Continuous-time LQR, infinite horizon.
Consider the system described by the equations 

\[\dot{x} = Ax + Bu, \quad y = Cx,\]

where

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

(a) Determine the optimal control \(u^*(t) = F^* x(t), t \geq 0\) which minimizes the performance index \(J = \int_0^\infty (y^2(t) + \rho u^2(t))dt\) where \(\rho\) is positive and real.

(b) Observe how the eigenvalues of the dynamic matrix of the resulting closed loop system change as a function of \(\rho\). Can you comment on the results?

Problem 5: Continuous-time LQR.
Consider an object of mass \(m = 1\) moving along the x-axis in response to a force input \(u(t)\). The object’s dynamics can be described simply as \(\ddot{x} = u(t)\). Suppose you would like to design an input \(u(t)\) which will
move the object from any initial position and velocity, to come to rest at the position $x = 4$. Using the linear quadratic regulator discussed in class, formulate an appropriate quadratic cost functional, and solve the problem in MATLAB, python or whatever you like, showing simulations of your results for different weightings on the state and input.