GOALS:
- present Hermitian : Unitary matrices
- present the Singular Value Decomposition (SVD).

REFS: Callier : Desoer
       Appendix A (2A.7)
Self-adjoint maps

Given \((H, \mathcal{F}, \langle \cdot, \cdot \rangle_H)\), a Hilbert space.

Let \(A : H \rightarrow H\) be a continuous linear map with adjoint \(A^* : H \rightarrow H\). We say that the map \(A\) is self-adjoint iff \(A = A^*\), or equivalently,

\[
\langle x, Ay \rangle_H = \langle Ax, y \rangle_H \quad \forall x, y \in H.
\]

**Example** (Hermitian matrices)

Let \(H = \mathbb{F}^n\) and let \(A\) be represented by a matrix \(A = (a_{ij})_{i,j \in \{1, \ldots, n\}} \in \mathbb{F}^{n \times n}\). Then \(A\) is self-adjoint iff the matrix \(A^*\) is Hermitian, or equivalently, \(A = A^*\), meaning \(a_{ij} = \overline{a_{ji}}\) \(\forall i, j \in \{1, \ldots, n\}\), or that \(A\) is equal to its complex conjugate transpose.

**Defn** Unitary matrix

A matrix \(U \in \mathbb{F}^{n \times n}\) is said to be unitary iff \(U^* U = U U^* = I_n\) (equivalently, the columns of \(U\) form orthonormal bases of \(\mathbb{F}^n\)). If \(\mathbb{F} = \mathbb{R}\), such a matrix is called orthogonal.
G. The Singular-Value Decomposition

1 Theorem Let \( M \in \mathbb{C}^{m \times n} \) with rank \( (M) = r \). Then we can find unitary matrices \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \) such that

\[
M = U \Sigma V^* = U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V^*
\]

where

\[
\Sigma_1 = \begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_r
\end{bmatrix}
\]

The real numbers \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \) are called the singular values of \( M \) and the representation above is called the singular-value decomposition of \( M \).

2 Theorem Let \( M \in \mathbb{C}^{m \times n} \) with rank \( (M) = r \) and let \( M = U \Sigma V^* \) be the singular-value decomposition of \( M \). Partition \( U \) and \( V \) as

\[
U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}
\]

where \( U_1 \) and \( V_1 \) are in \( \mathbb{C}^{r \times r} \). Then

(a) The columns of \( U_1 \) and \( U_2 \) form orthonormal bases for \( \mathcal{R}(M) \) and \( N(M^*) \) respectively.

(b) The columns of \( V_1 \) and \( V_2 \) form orthonormal bases for \( \mathcal{R}(M^*) \) and \( N(M) \) respectively.

3 Computing the singular-value decomposition

While definitely not the method of choice vis-a-vis numerical aspects, the following result provides an adequate method for determining the singular-value decomposition of a matrix.

Theorem Let \( M \in \mathbb{C}^{m \times n} \) with rank \( (M) = r \). Let \( \lambda_1, \ldots, \lambda_r \) be the nonzero eigenvalues of \( M^* M \). These will be nonnegative because \( M^* M \geq 0 \). Also, from item (F.4) it follows that there exist unitary matrices \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \) such that

\[
MM^* = U \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad M^* M = V \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} V^*
\]

where \( \Lambda = \text{diag} (\lambda_1, \ldots, \lambda_r) \).
Then the singular-value decomposition of $M$ is

$$M = U \left[ \begin{array}{cc} \Lambda^{\frac{1}{2}} & 0 \\ 0 & 0 \end{array} \right] V^*$$

and the singular values of $M$ are $\lambda_1^{\frac{1}{2}}, \ldots, \lambda_r^{\frac{1}{2}}$.

4 Optimal rank $q$ approximations of matrices

**Theorem**  Let $M \in \mathbb{C}^{m \times n}$ with rank $(M) = r$ have singular value decomposition as

$$M = U \left[ \begin{array}{cc} \Sigma_1 & 0 \\ 0 & 0 \end{array} \right] V^*$$

where $\Sigma_1 = \left[ \begin{array}{ccc} \sigma_1 & 0 & \cdots \\ 0 & \sigma_2 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots \sigma_r \end{array} \right]$.

Define the matrix

$$\hat{M} = U \left[ \begin{array}{cc} \hat{\Sigma}_1 & 0 \\ 0 & 0 \end{array} \right] V^*$$

where $\hat{\Sigma}_1 = \left[ \begin{array}{ccc} \sigma_1 & 0 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots \sigma_q \end{array} \right]$. 