EECS 221A LECTURE 7

**Goals of this lecture:**

- Proof of existence and uniqueness of solutions to
  \[ \dot{x} = f(x, t), \quad x(t_0) = x_0. \]

- Piecewise continuity
- Lipschitz continuity
- Cauchy sequence
- Banach space
- Bellman-Grönwall Lemma
- Examples

**Refs:**

Callier & Desoer,
Appendix B,
GB.1.
Differential Equations

\[ \dot{x} = f(x,t); \quad x(t_0) \in \mathbb{R}^n; \quad x(t_0) = x_0 \]
\[ f(x,t): \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \]

Under what conditions

(a) does a solution exist, i.e., meaning that \( x(t_0) = x_0 \) guarantees that \( x(t) \) is defined for all \( t \geq t_0 \) ?

(b) is the solution unique?

**Def.** \( f(x,t): \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \) is piecewise continuous in \( t \) \( \forall x \) if \( f(x,\cdot): \mathbb{R}_+ \to \mathbb{R}^n \) is continuous except at points of discontinuity, and there can only be finitely many points of discontinuity in any compact interval.

**Def.** \( f(x,t): \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \) is Lipschitz continuous in \( x \) \( \forall t \) if there exists a piecewise continuous function \( k(\cdot): \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[ \| f(x,t) - f(y,t) \| \leq k(t) \| x - y \| \]

This inequality is called the Lipschitz condition.

\( \forall x, y \in \mathbb{R}^n. \quad \forall t \in \mathbb{R}_+ \)
Fundamental Theorem of Differential Equations

Consider \( \dot{x} = f(x,t) \), \( x(t_0) = x_0 \), with \( f(x,t) \) piecewise continuous in \( t \) and lipschitz continuous in \( x \). Then there exists a unique function of time \( \phi(t) : \mathbb{R}_+ \to \mathbb{R}^n \) which is \( C^1 \) almost everywhere satisfying:

\[
\phi(t_0) = x_0
\]

\[
\dot{\phi}(t) = f(\phi(t), t) \quad \forall t \in [t_0, t] \quad D
\]

where \( D \) is the set of discontinuity points for \( f \) as a function of \( t \).

Using this, we will be able to generate mathematical models of input-output systems.

Consider, for example,

\[
\dot{x} = f(x, u, t), \quad f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n
\]

\[
y = h(x, u, t), \quad h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m
\]

If \( f \) is lipschitz continuous in \( x \), continuous in \( u \), and piecewise continuous in \( t \), and if \( u(\cdot) \) is piecewise continuous in \( t \), we are guaranteed that given \( x(t_0) = x_0 \) \( \exists ! x(t) \in \mathbb{R}^n \) satisfying the differential equation. With this, \( \exists ! y(\cdot) \in \mathbb{R}^m \) called the output of the system.
Note: if the Lipschitz condition does not hold, it may be that the solution cannot be continued beyond a certain time:

**Example:** consider

\[ \xi(t) = \xi(t)^2, \quad \xi(0) = \frac{1}{c}, \quad c \neq 0 \]

where \( \xi(t) : \mathbb{R}^+ \rightarrow \mathbb{R} \).

This differential equation has the solution \( \xi(t) = \frac{1}{c-t} \) on \( t \in (-\infty, c) \).

When \( t \rightarrow c \), \( ||\xi(t)|| \rightarrow \infty \)

"finite escape time at \( c \)."

**Proof (of the Fundamental Theorem)**

(1) Construct a sequence of continuous functions

\[ x_{m+1}(t) := x_0 + \int_{t_0}^{t} f(x_m(r), r) \, dr \]

where \( x_0(t_0) = x_0 \) and \( m = 0, 1, 2, \ldots \)

The idea is to show that the sequence of continuous functions

\[ \{x_m(\cdot)\}_{m=0}^{\infty} \]

converges to

(1) a continuous function \( \phi(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^n \)

(2) which is a solution of \( \dot{x} = f(x, t) \)

\[ x(t_0) = x_0 \]

"construction of a solution by iteration"
To show (1), we show that $\xi x_m(\cdot)_{t_0} \to _t \infty$ is a Cauchy sequence in a Banach space $(C([t_1, t_2], \mathbb{R}^n), \mathbb{R}, \| \cdot \|_\infty)$, where $t_0 \in [t_1, t_2]$:

$$\| x_{m+1}(t) - x_m(t) \| = \| \int_t^{t_0} \| f(x_m(\tau), \tau) - f(x_{m-1}(\tau), \tau) \| d\tau \|$$

$$\leq \int_t^{t_0} \| f(x_m(\tau), \tau) - f(x_{m-1}(\tau), \tau) \| d\tau$$

$$\leq \int_t^{t_0} k(\tau) \| x_m(\tau) - x_{m-1}(\tau) \| d\tau$$

Letting $\bar{K} = \sup_{[t_1, t_2]} K(t)$, then for $m = 1, 2, \ldots, \forall t \in [t_1, t_2]$,

$$\| x_{m+1}(t) - x_m(t) \| \leq \bar{K} \int_t^{t_0} \| x_m(\tau) - x_{m-1}(\tau) \| d\tau$$

Now, we know by the definition of $\xi x_m(\cdot)_{t_0} \to _t \infty$ that

$$x_1(t) := x_0 + \int_t^{t_0} f(x_0, \tau) d\tau , \quad t \in [t_1, t_2]$$

$$\therefore \| x_1(t) - x_0 \| \leq \int_t^{t_0} \| f(x_0, \tau) \| d\tau \leq \int_t^{t_1} \| f(x_0, \tau) \| d\tau =: M$$

Since $x_0$ is specified, $M$ is known.

$$\therefore \| x_2(t) - x_1(t) \| \leq M \bar{K} |t - t_0|$$
and \[ \| x_3(t) - x_2(t) \| \leq M \frac{K^2 H - t_0}{2!} \]

\[ \| x_{m+1}(t) - x_m(t) \| \leq M \frac{(K(t-t_0))^m}{m!} \]

now recalling that \[ \| f(\cdot) \|_\infty = \max_{t \in [t_1, t_2]} \| f(t) \| \]
and defining \[ T = t_2 - t_1 \]
\[ \| x_{m+1}(\cdot) - x_m(\cdot) \|_\infty \leq M \frac{(KT)^m}{m!}, m = 0, 1, 2 \ldots \]

To see that \( \{ x_m(\cdot) \}_{n=0}^\infty \) is a Cauchy sequence in \( (C([t_1, t_2], \mathbb{R}^n), \mathbb{R}, \| \cdot \|_\infty) \):

\[ \| x_{m+p}(\cdot) - x_m(\cdot) \|_\infty = \| \sum_{k=0}^{p-1} (x_{m+k+1}(\cdot) - x_{m+k}(\cdot)) \|_\infty \]
\[ \leq \sum_{k=0}^{p-1} \| x_{m+k+1}(\cdot) - x_{m+k}(\cdot) \|_\infty \]
\[ \leq M \sum_{k=0}^{p-1} \frac{(KT)^{m+k}}{(m+k)!} \]
\[ \leq M \frac{(KT)^m}{m!} \sum_{k=0}^{p-1} \frac{(KT)^k}{K!} \]

\[ \text{(since } (m+k)! \geq m! K!) \]
\[ \leq M \frac{(KT)^m}{m!} e^{KT} \]

\[ \text{(since } e^{KT} = \sum_{k=0}^{\infty} \frac{(KT)^k}{K!} \)]

\[ \therefore \{ x_m(\cdot) \}_{n=0}^\infty \text{ is Cauchy} \]
To show (2) \( \varphi(\cdot) \) is a solution of the d.e.:

\[
x_{m+1}(t) = x_0 + \int_{t_0}^{t} f(x_m(r), r) \, dr
\]

as \( m \to \infty \), \( x_m(\cdot) \to \varphi(\cdot) \) (on \( [t, t_{2}] \))

we've just proved \( \Phi_\infty \)

\[
\therefore \text{need to show} \quad \int_{t_0}^{t} f(x_m(r), r) \, dr \to \int_{t_0}^{t} f(\varphi(r), r) \, dr
\]

as \( m \to \infty \).

Indeed,

\[
\begin{align*}
\| \int_{t_0}^{t} (f(x_m(r), r) - f(\varphi(r), r)) \, dr \| &
\leq \int_{t_0}^{t} k(r) \| x_m(r) - \varphi(r) \| \, dr \quad \text{by Lipschitz} \\
&
\leq K \| x_m(\cdot) - \varphi(\cdot) \|_\infty \cdot T \\
&
\leq K M e^{|ET|} \left( \frac{|ET|^m}{m!} \right) \cdot T \quad \text{(by letting} \ p \to \infty \ \text{in (**) )}
\end{align*}
\]

\[
\therefore \varphi(t) = x_0 + \int_{t_0}^{t} f(\varphi(r), r) \, dr \quad \forall t \in [t_1, t_2]
\]

\[
\therefore \dot{\varphi}(t) = f(\varphi(t), t) \quad \forall t \in [t, t_2], t \notin D
\]

Since \( [t, t_2] \) is arbitrary (containing \( t_0 \)) then we can conclude that the proposed iterative scheme converges to a soln \( \varphi \) on \( \mathbb{R}_+ \).
We have constructed a solution on $\mathbb{R}_+$. Conceivably, a different construction might lead to another solution. Thus we have to verify that $\emptyset$ is the unique solution:

(Uniqueness).

To prove uniqueness, we will need the Bellman–Gronwall lemma.

Let $u(\cdot)$, $k(\cdot)$ be real-valued, piecewise continuous functions on $\mathbb{R}_+$; and assume $u(\cdot)$, $k(\cdot) > 0$ on $\mathbb{R}_+$. Assume $c_i > 0$, $t_0 \in \mathbb{R}_+$.

Then, if

$$u(t) \leq c_i + \int_{t_0}^{t} k(\tau) u(\tau) d\tau \quad (***)$$

Then

$$u(t) \leq c_i e^{\int_{t_0}^{t} k(\tau) d\tau}$$

Proof: WLOG assume $t > t_0$.

Let $U(t) = c_i + \int_{t_0}^{t} k(\tau) u(\tau) d\tau$.

Thus $u(t) \leq U(t)$ (***)

Multiply both sides of (***) by the non-negative function $k(t) e^{-\int_{t_0}^{t} k(\tau) d\tau}$.

resulting in:

$$\frac{d}{dt} \left\{ U(t) e^{-\int_{t_0}^{t} k(\tau) d\tau} \right\} \leq 0$$

and thus $u(t) \leq U(t) \leq c_i e^{-\int_{t_0}^{t} k(\tau) d\tau}$

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Finally, using Bellman–Gronwall to show $9$. uniqueness:
$$\dot{x} = f(x(t), t), \quad x(t_0) = x_0$$

$f$ is:
- piecewise cont's in $t$
- Lipschitz cont's in $x$

We have shown there exists a solution $\phi(t)$ to the above; suppose there are two solutions $\phi \neq \psi$ satisfying the above:

$$\phi(t) = f(\phi(t), t), \quad \phi(t_0) = x_0$$
$$\psi(t) = f(\psi(t), t), \quad \psi(t_0) = x_0$$

$$\therefore \phi(t) - \psi(t) = \int_{t_0}^{t} (f(\phi(r), r) - f(\psi(r), r)) \, dr$$

$$\forall t \in \mathbb{R}^+$$

$$\therefore ||\phi(t) - \psi(t)|| \leq K \int_{t_0}^{t} ||\phi(r) - \psi(r)|| \, dr \quad \forall t \in [t_1, t_2]$$

From Bellman–Gronwall:

if $||\phi(t) - \psi(t)|| \leq C_1 + K \int_{t_0}^{t} ||\phi(r) - \psi(r)|| \, dr$

Then $||\phi(t) - \psi(t)|| \leq C_1 e^{K(t-t_0)}$

but here, $C_1 = 0$, thus $||\phi(t) - \psi(t)|| = 0$

$\Rightarrow \phi(t) = \psi(t)$
Example:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
x(t_0) &= x_0
\end{align*}
\]

Show the solution is unique.

Solution: Assume \( \phi(t), \gamma(t) \) are two solutions.

\[
\therefore \phi(t_0) = \gamma(t_0) = x_0
\]

\[
\begin{align*}
\phi(t) &= A(t)\phi(t) + B(t)u(t) \\
\gamma(t) &= A(t)\gamma(t) + B(t)u(t)
\end{align*}
\]

\[
\therefore \phi(t) - \gamma(t) = \int_{t_0}^{t} (A(t)\phi(t) - A(t)\gamma(t)) \, dt
\]

\[
\therefore \| \phi(t) - \gamma(t) \| \leq \| A(t) \|_{\infty, [t_0, t]} \int_{t_0}^{t} \| \phi(t) - \gamma(t) \| \, dt
\]

\[
\therefore \text{by Bellman-Gronwall,}
\]

\[
\| \phi(t) - \gamma(t) \| \leq C_1 + \| A(t) \|_{\infty, [t_0, t]} \int_{t_0}^{t} \| \phi(t) - \gamma(t) \| \, dt
\]

\[
\Rightarrow \| \phi(t) - \gamma(t) \| \leq C_1 e^{\| A(t) \|_{\infty, [t_0, t]} (t-t_0)}
\]

This is true \( \forall C_1 > 0 \), so set \( C_1 = 0 \) . . .
**Example** Reverse-time differential equation.

Consider \( x = f(x, t) \), \( x(t_0) = x_0 \).

Suppose \( f(x, t) \) is such that the solution exists and is unique for \( t > t_0 \).

Now consider \( \tau < t_0 \), \( \tau = -(t - t_0) \).

We want \( z(\tau) = x(t) \) if \( t < t_0 \).

\[
\frac{d}{d\tau} z(\tau) = -\frac{d}{dt} x(t) = -f(x(t), t) = -f(z(\tau), t_0 - \tau) =: f(z(\tau), \tau)
\]

\( f \) lipschitz \( \Rightarrow \) \( f \) lipschitz. Why?

\[
\therefore \frac{d}{d\tau} z(\tau) = f(z, \tau) \quad z(0) = x_0
\]

\( \therefore \) in the reverse time d.e., solution exists and is unique.

**Exercise:** \( \dot{x} = f(x) \), \( x(t_0) = x_0 \), lipschitz \( \Rightarrow \) soln exists and is unique.

This can't happen: \( x \longrightarrow x_0 \)

Can this? : \( x \longrightarrow x_0 \), \( x_0 \longrightarrow x_{02} \)