Goals of this Lecture:

- define the adjoint linear system (dual system representation)
- pairing lemma
- Linear Quadratic Optimization
  - necessary and sufficient conditions
  - the optimal input
  - two point boundary value problem
  - optimal cost
  - Hamiltonian system viewpoint
  - Riccati D.E. viewpoint
  - examples

Refs: Callier & Desoer §2.1.6, 2.1.7, 2.2.3
Adjoint equation

Consider the time-varying linear system without input:

\[ \dot{x}(t) = A(t) x(t) \quad t \in \mathbb{R}_+, \quad x(t) \in \mathbb{R}^n \]
\[ A(\cdot) \in \text{PC}(\mathbb{R}_+, \mathbb{R}^{n \times n}) \]

with state transition matrix:
\[ \Phi(t, t_0) = A(t) \Phi(t, t_0) \]

To this equation (1), we associate the adjoint differential equation:

\[ \dot{\tilde{x}}(t) = -A^*(t) \tilde{x}(t) \quad t \in \mathbb{R}_+, \quad \tilde{x}(t) \in \mathbb{R}^n \]
\[ -A^*(\cdot) \in \text{PC}(\mathbb{R}_+, \mathbb{R}^{n \times n}) \]

with state transition matrix:
\[ \tilde{\Phi}(t, t_0) = -A^*(t) \tilde{\Phi}(t, t_0) \]

In (2), \( A^* \) is the Hermitian transpose of \( A \) (if \( A \) is real this is simply the transpose of \( A \)).

Now recall that \( (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \) is a Hilbert space with inner product:
\[ \langle x, y \rangle = x^* y \quad \forall x, y \in \mathbb{R}^n. \]

(* from lecture note 4).
Thus for any matrix $A \in \mathbb{R}^{n \times n}$

$$< y, Ax > = y^* Ax = < A^* y, x > \quad \forall x, y \in \mathbb{R}^n$$

gives us the defining relation for $A^* \in \mathbb{R}^{n \times n}$, the adjoint of $A$ as a linear map.

**Fact** \[ \mathcal{I}^* (t, t_0) = \mathcal{I} (t_0, t) \]

**Proof** Need to prove that:

\[ \mathcal{I}^* (t, t_0) \mathcal{I} (t, t_0) = I \]

* recall why, from properties of state transition matrices.

\[
\frac{d}{dt} \text{LHS} (t) = \mathcal{I}^* (t, t_0) \mathcal{I} (t, t_0) + \mathcal{I}^* (t, t_0) \mathcal{I} (t, t_0)
\]

\[
= -\mathcal{I}^* (t, t_0) \mathcal{A} (t) \mathcal{I} (t, t_0) + \mathcal{I}^* (t, t_0) \mathcal{A} (t) \mathcal{I} (t, t_0)
\]

\[ = 0 \]

and \( \text{LHS} (t_0) = I \).

\[
\frac{d}{dt} \text{RHS} (t) = 0
\]

and \( \text{RHS} (t_0) = I \)

Hence, \( \text{LHS} (t) = \text{RHS} (t) \) \[ \quad \square \]
Now consider the system representation denoted \( R(\cdot) = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)] \) given by:
\[
\begin{align*}
\dot{x}(t) &= A(t) x(t) + B(t) u(t) \quad x(t) \in \mathbb{R}^n, \\
y(t) &= C(t) x(t) + D(t) u(t) \quad y(t) \in \mathbb{R}^n.
\end{align*}
\]

The \underline{dual system representation}, denoted \( \tilde{R}(\cdot) = [-A^*(\cdot), -C^*(\cdot), B(\cdot)^*, D^*(\cdot)] \) given by:
\[
\begin{align*}
-\dot{x}(t) &= A^*(t) \tilde{x}(t) + C^*(t) \tilde{u}(t) \quad \tilde{x}(t) \in \mathbb{R}^n, \\
\tilde{y}(t) &= B^*(t) \tilde{x}(t) + D^*(t) \tilde{u}(t) \quad \tilde{y}(t) \in \mathbb{R}^n.
\end{align*}
\]

\underline{Lemma} \hspace{1em} (Pairing between \( R(\cdot) \) and \( \tilde{R}(\cdot) \))

\( \tilde{R}(\cdot) \) is related to \( R(\cdot) \) as follows:
\[
\begin{align*}
\langle \tilde{x}(t), x(t) \rangle + \int_{t_0}^{t} \langle \tilde{u}(\tau), y(\tau) \rangle \, d\tau \\
= \langle \tilde{x}(t_0), x(t_0) \rangle + \int_{t_0}^{t} \langle \tilde{y}(\tau), u(\tau) \rangle \, d\tau
\end{align*}
\]

\( \forall t, t_0 \in \mathbb{R}_+ \)
\( \forall (x(t_0), u(\cdot)) \in \mathbb{R}^n \times U \)
\( \forall (\tilde{x}(t), \tilde{u}(\cdot)) \in \mathbb{R}^n \times \tilde{U} \)

\[ \textbf{Proof:} \hspace{1em} \text{exercise.} \]
Linear Quadratic Optimization

Given \( \dot{x}(t) = A(t)x(t) + B(t)u(t) \) \( t \in [t_0, t, J] \), \( x(t_0) = x_0 \)

Minimize a quadratic cost functional:

\[
J(u(\cdot)) := \frac{1}{2} \int_{t_0}^{t_1} \left[ \|u(t)\|^2 + \|C(t)x(t)\|^2 \right] dt + x(t_1)^* S x(t_1)
\]

where \( C(\cdot) \in PC(R_+, R^{n_0}) \) and \( S \in R^{n \times n} \), \( S = S^* = 0 \)

ie \( J(\cdot) : PC([t_0, t, J, R^{n_0}]) \to R : u(\cdot) \to J(u(\cdot)) \)

Thus, we would like to solve:

\[
\min \{ J(u(\cdot)) : u(\cdot) \in PC \}
\]

Analysis

Let \( u \in PC \) be any input and \( S u \in PC \) any perturbation. Expanding \( J(u + \varepsilon S u) \) about \( \varepsilon = 0 \) yields:

\[
J(u + \varepsilon S u) = J(u) + \varepsilon \dot{S}T(Su) + o(\varepsilon)
\]

Claim For \( u \) to be a minimizer of \( J \)

(meaning \( \forall S u \in PC, \forall \varepsilon \in R, J(u + \varepsilon S u) \leq J(u) \))

it is necessary and sufficient that

\( \dot{S}T(Su) = 0 \) \( \forall S u \in PC \).
Proof (necessity) if $u$ is a minimizer, then
\[ \forall \epsilon \in \mathbb{R}, \epsilon \left[ Sf(su) + \frac{o(\epsilon)}{\epsilon} \right] = J(u + \epsilon su) - J(u) \geq 0. \]

:. if $\epsilon > 0$, $\epsilon \to 0$ then $Sf(su) \geq 0 \Rightarrow Sf(su) = 0.$
if $\epsilon < 0$, $\epsilon \to 0$ then $Sf(su) \leq 0 \Rightarrow Sf(su) = 0.$

(sufficiency)
The state transition map of $R(\cdot)$ is linear in $(x_0, u)$, thus any perturbed input $u + \epsilon su$ $(x_0$ fixed) generates a perturbed state trajectory $x + \epsilon sx$

Hence, using (4):
\[ J(u + \epsilon su) = J(u) + \epsilon Sf(su) + \epsilon^2 T(su) \]

where
\[ Sf(su) = \int_{t_0}^{t_1} <u, su> dt + \int_{t_0}^{t_1} <c^*c x, dx> dt \]

and $sx(\cdot)$ is related to $su(\cdot)$ by
\[ sx = A sx + B su \]
\[ sx_0 = 0 \]

and (from (4)) $T(su) \geq 0 \forall su \in PC.$

Hence if $u$ is such that $Sf(su) = 0 \forall su \in PC$

then (using $\epsilon = 1$) $J(u) = J(u + su).$
Determination of the optimal input $u$:

Consider the dynamics of the perturbation:

$$\dot{x} = Ax + Bu, \quad x_0 = 0$$

and its dual:

$$-\dot{\tilde{x}} = A^*\tilde{x} + \tilde{u}$$

we have by the pairing lemma that:

$$(6) \quad <\tilde{x}_1, Sx_1> + \int_{t_0}^{t_1} <\tilde{u}, Sx> dt = <\tilde{x}_0, Sx_0> + \int_{t_0}^{t_1} B^*\tilde{x}, Su> dt$$

Now consider (5) and (6):

if we choose $\tilde{u} = C_1^* C_1 \tilde{x}$

$$\tilde{x}_1 = Sx_1$$

then $ST(Su) = \int_{t_0}^{t_1} <u + B^*\tilde{x}, Su> dt$

giving us an explicit representation for $ST(Su)$ as a function of $Su$.

Also, consider a choice of $Su = u + B^*\tilde{x}$. With this, $ST(Su) = 0$ iff $u(t) = -B^*(t)\tilde{x}(t)$ $\forall t \in [t_0, t_1]$.

(* note that this is a fine choice since we've just shown that a necessary, sufficient condition for optimality is $ST(Su) = 0$ $\forall Su \in PC$).
This derivation results in the following optimality characterization:

**Theorem (Solution of the Linear Quadratic Optimization problem).**

The solution of the problem \( \min \mathbb{E} J(u(\cdot)) : u(\cdot) \in PC \), where \( J(u(\cdot)) \) is given by (4) and \( x(\cdot) \) as a function of \( u(\cdot) \) given by (3), is given by the optimal input:

\[
u(t) = -B^*(t)\tilde{x}(t), \quad t \in [t_0, t_f]
\]

where \( \tilde{x}(\cdot) \in C([t_0, t_f], \mathbb{R}^n) \) is defined as the partial state trajectory of the 2n-dimensional two point boundary value problem given by

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) - B(t)B^*(t)\tilde{x}(t) \\
\dot{\tilde{x}}(t) &= -C^*(t)C(t)x(t) - A^*(t)\tilde{x}(t)
\end{align*}
\]

where \( x(t_0) = x_0 \)

and \( \tilde{x}(t_f) = Sx(t_f) \)

Aside (7) is a Hamiltonian system with Hamiltonian matrix \( H(t) = [A(t), -B(t)B^*(t)] \):

\[
\begin{bmatrix}
A(t) & -B(t)B^*(t) \\
-C^*(t)C(t) & -A^*(t)
\end{bmatrix}
\]
Now let's compute the *optimal cost*:

recall substitutions:

\[ u(t) = -B^*(t) \tilde{x}(t) \]

\[ \tilde{u}(t) = C^*(t) C(t) x(t) \]

with \( \tilde{x}(t_1) = S x(t_1) \)

Then (7) can be rewritten as

\[ \dot{x}(t) = A(t) x(t) + B(t) u(t) \]

\[ -\dot{x}(t) = A^*(t) \tilde{x}(t) + \tilde{u}(t) \]

such that \( x(t_0) = \chi_0 \)

\( \tilde{x}(t_1) = S x(t_1) \)

By the pairing lemma,

\[ \langle \tilde{x}(t_0), \chi_0 \rangle = \langle \tilde{x}(t_1), x(t_1) \rangle + \int_{t_0}^{t_1} \langle \tilde{u}, x \rangle \, dt - \int_{t_0}^{t} \langle B^* \tilde{x}, u \rangle \, dt \]

\[ = \langle S x(t_1), x(t_1) \rangle + \int_{t_0}^{t_1} \| C x \|^2 \, dt + \int_{t_0}^{t} \| u \|^2 \, dt \]

\[ = 2 J(u(\cdot)) \]

\[ \Rightarrow J(u(\cdot)) = \frac{1}{2} \langle \tilde{x}(t_0), \chi_0 \rangle \]
Theorem [Optimal LQ state feedback by the Hamiltonian system]

Consider the LQ optimization problem given by
\[ \min \mathcal{J}(u(.)) : u(.) \in \mathcal{PCF} \]
described by (3), (4).
Suppose the horizon \( t_f \) is fixed and to \( \in [0, t_f) \)
is arbitrary. Then:

(i) The LQ problem is solved by the fixed linear state feedback law:
\[ u(t) = -B^*(t) \tilde{x}(t) X(t)^{-1} x(t), \quad t \in [0, t_f] \]
where \( X(.) \) and \( \tilde{X}(.) \) are \( nxn \) real matrix-valued functions with \( \det X(t) \neq 0 \) and are determined as the unique solution of the backwards Hamiltonian linear matrix d.e.:
\[ \begin{aligned}
\frac{d}{dt} \begin{bmatrix} X(t) \\ \tilde{X}(t) \end{bmatrix} &= \begin{bmatrix} A(t) & -B(t)B^*(t) \\ -C^*(t)C(t) & -A^*(t) \end{bmatrix} \begin{bmatrix} X(t) \\ \tilde{X}(t) \end{bmatrix} \\
&= H(t) \begin{bmatrix} X(t) \\ \tilde{X}(t) \end{bmatrix}, \quad t \in [0, t_f]
\end{aligned} \]

(8)

with \( X(t_f) = I \)
and \( \tilde{X}(t_f) = S \).

(ii) The LQ problem has the optimal cost \( J_0 \) given by the quadratic form
\[ J_0 = \frac{1}{2} \langle \tilde{x}(t_0), X(t_0)^{-1} x_0, x_0 \rangle \]
and generate the optimal closed loop system dynamics described by the linear homogeneous d.e.:

\[
\dot{x}(t) = [A(t) - B(t)B^*(t)\tilde{x}(t)X(t)^{-1}]x(t) \quad t \in [t_0, t_f]
\]

with \( x(t_0) = x_0 \).

Remark

Denoing \( F(t) := B^*(t)\tilde{x}(t)X(t)^{-1} \in \mathbb{R}^{n \times n} \),

the optimal control is to modify the open loop dynamics by a linear time-varying state feedback law:

\[
U(t) = -F(t)x(t)
\]

Proof of Theorem

Step 1 first show that the partial solution \( x(t) \in \mathbb{R}^{n \times n} \) of (8) is non-singular for all \( t \in [0, t_f] \).

(see Callier & Desoer, p36).

Step 2 (Theorem holds)

let \( (\tilde{x}(-), \tilde{x}(\cdot)) \) be the backwards solution of (8). Since \( \det x(t) \neq 0 \), for a given initial
condition $x(t_0) = x_0$, there exists a unique vector $k \in \mathbb{R}^n$ s.t at $t_0$, $x_0 = X(t_0)k$.

Hence it follows that $x(t) := X(t)k$ 
$\tilde{x}(t) := \tilde{X}(t)k$.

is the unique solution to the Hamiltonian system (7).

This implies that $k = X(t_0)^{-1}x_0 = X(t)^{-1}x(t)$

Hence, by the general theorem on p. 8, the solution of the CL optimization problem is given by:

$$u(t) = -B^*(t)\tilde{x}(t)$$
$$= -B^*(t)\tilde{X}(t)k$$
$$= -B^*(t)\tilde{X}(t)X(t)^{-1}x(t)$$

(i)

Now, the optimal cost is given by:

$$2J(u(\cdot)) = <\tilde{x}(t_0), x_0>$$
$$= <\tilde{X}(t_0)k, x_0>$$
$$= <\tilde{X}(t_0)X(t_0)^{-1}x_0, x_0>$$

(ii)
An equivalent statement of the previous theorem is:

**Theorem** [Optimal LQ state feedback by the Riccati d.e.]

Consider the LQ optimization problem given by
\[
\min \{ J(u(t)) : u \in P \}\]
described by (3), (4), with the horizon \( t \),
fixed and \( t \in [0, t_f] \) arbitrary. Then:

(i) The LQ problem is solved by the fixed linear state feedback law:
\[
u(t) = -B^*(t)P(t)x(t), \quad t \in [0, t_f]
\]
where \( P(t) = P^*(t) \geq 0 \) is the \( n \times n \) real matrix valued function defined on \( [0, t_f] \) as the unique solution of the matrix d.e.:

\[
-\dot{P}(t) = A^*(t)P(t) + P(t)A(t) - P(t)B(t)B^*(t)P(t) + C^*(t)C(t)
\]
(10)

With \( P(t_f) = S = S^* \geq 0 \)

[10] is called the **matrix Riccati equation**

(ii) The LQ problem has the optimal cost \( J \) given by the quadratic form:
\[
J_0 = \frac{1}{2} < P(t_0)\mathbf{x}_0, \mathbf{x}_0 >
\]
and generates the optimal closed loop system dynamics, given by:
\[
\dot{x}(t) = \left[ A(t) - B(t)B^*(t)P(t) \right] x(t)
\]
\[ X(\tau) = X_0. \]

**Sketch of Proof (for details see Callier & Desoer p38).**

On comparing the statement of this theorem with the previous, we are done if the solution of (10) is

\[ P(t) = \tilde{X}(t) \cdot X(t)^{-1} \]

where \((X(t), \tilde{X}(t))\) is the solution of (8).

Assume

\[ P(t) = \tilde{X}(t) \cdot X(t)^{-1} \]

\[ \therefore \quad \dot{P}(t) = \tilde{X}(t) X(t)^{-1} - \tilde{X}(t) X^{-1} \ddot{X} X^{-1} \]

(using (8))

\[ = \left[ -C^* C X - A^* \tilde{X} J X^{-1} - P[A X - B B^* \tilde{X}] X^{-1} \right] \]

\[ = \left[ -C^* C - A^* P - PA + P B B^* P \right] \]

where \( P(t_1) = \tilde{X}(t_1) X(t_1)^{-1} = S. \)

**Summary**

1. For fixed horizon \( t_1 \), the LQ optimization problem on any \([t_0, t_1]\) is solved by computing:

\[ F(t) = B(t)^* P(t) \]

to generate the optimal input \( x(t) = -F(t)x(t) \).

2. The nxn matrix \( P(t) \) can either be computed through (8), then

\[ P(t) = \tilde{X}(t) \cdot X(t)^{-1}, \]

or by solving the backwards Riccati d.e. (10).
EXAMPLE 1:

Given \( \dot{x}(t) = f(x(t), u(t), t) \)
\[ x(t_0) = x_0 \]
and given a cost functional \( J(u(\cdot)) \) given by
\[ J(u(\cdot)) = g(x(t_1)) = g(s(t_1, t_0, x_0, u(\cdot))) \]
Find \( u \in PC \) that solves \( \min \mathcal{E} J(u(\cdot)) : u \in PC \).

Solution: in general, no closed form solution,
yet in practice we'd start from a reasonable
guess and try to improve on it: let \( u_0(\cdot), x_0(\cdot) \)
be an initial guess and perturb it to
\( u_0(\cdot) + \delta u(\cdot), x_0(\cdot) + \delta x(\cdot) \). The problem
is now to find \( \delta u(\cdot) \) to decrease
\[ g(x_0(t_1), s(t_1, t_0, x_0, u_0(\cdot))) < g(x_0(t_1)) \].

We know that the dynamics of the perturbation
in \( x \) are:

\[ \begin{align*}
\dot{s}x(t) &= A(t) \delta x(t) + B(t) \delta u(t) \\
\delta x(t_0) &= 0
\end{align*} \]

where
\[ A(t) := D_1 f(x_0, u_0, t) \]
\[ B(t) := D_2 f(x_0, u_0, t) \]

let \( \Phi(t, t') \) be the state transition matrix
associated with (11).
Then \( S x(t, \tau) = \int_{t_0}^{t_1} \Phi(t, t, \tau) B(t, \tau) S u(t, \tau) d\tau \)

also \( S g |_{t_1} = D g |_{x_0(t_1)} : S x(t, \tau) = A^T S x(t, \tau) \)

Thus \( S g |_{t_1} = A^T \int_{t_0}^{t_1} \Phi(t, t, \tau) B(t, \tau) S u(t, \tau) d\tau \)

\( = \int_{t_0}^{t_1} [A^T \Phi(t, t, \tau) B(t, \tau)] S u(t, \tau) d\tau \)

Now define \( \dot{x} = -A^*(t) \tilde{x}(t), \tilde{x}(t, \tau) = \lambda \)

Thus \( \tilde{x}(t) = \Phi(t, t_1) \lambda \)

and \( S g |_{t_1} = \int_{t_0}^{t_1} \tilde{x}^T(t) B(t, \tau) S u(t, \tau) d\tau \)

If \( B^T(t) p(t) \neq 0 \), choose \( S u(t) = -\alpha B^T(t) \tilde{x}(t) \)

for a small enough
\( g(x_0(t_1) + S x(t_1)) < g(x_0(t_1)) \)

If \( B^T(t) p(t) = 0 \), then a local optimum has been reached. Thus the necessary condition for reaching a local optimum is:
\( \dot{x}_0 = f(x_0, u_0, t), x_0(t_0) = x_0 \)
\( \dot{x} = -D_i f(x_0(t), u_0(t), t) \tilde{x} \)
\( \tilde{x}(t_1) = D g |_{x_0(t_1)} \)

with \( D_z f(x_0, u_0, t) \tilde{x}(t) = 0 \).
Alternately, define
\[ H(x, u, p, t) = \bar{x}^T f(x, u, t) \]
Choose \( u_0(t) \) so as to locally minimize \( H(x, u, \bar{x}, t) \)
1e, so that \( \bar{x}^T D_2 f(x, u, t) = 0 \).
Then, define
\[
\begin{cases}
\dot{x} = \left( \frac{\partial H}{\partial \bar{x}} \right)^T ; & x(t_0) = x_0 \\
\dot{x} = -\left( \frac{\partial H}{\partial x} \right)^T ; & x(t_1) = D, g^T(x(t_1)).
\end{cases}
\]
\[ \Rightarrow \text{Hamilton's equations.} \]
EXAMPLE 2: Time-Invariant Linear Systems

In his 1960 paper, R.E. Kalman posed and solved the linear quadratic optimal control problem which we now describe. Let us suppose we are given a controllable state space description

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

and suppose also that we would like to control the input to the system in a way that minimizes the quadratic integral:

\[ J = \int_{0}^{\infty} (y^T Q y + u^T R u)(t) \, dt \]

in which the matrices \( Q \) and \( R \) are chosen by the designer.

Rewriting \( J \):

\[ J = \int_{0}^{\infty} y^T(t) Q y(t) \, dt + \int_{0}^{\infty} u^T(t) R u(t) \, dt \]
The first term in \( J \) above is a measure of the energy in the output, while the second term is the weighted control energy. Thus, the objective is therefore to find a \( u(t) \) such that \( J \) is as small as possible. In order for both terms in \( J \) to be non-negative, we assume that \( R \) is positive definite and \( Q \) is positive semi-definite.

In practice, \( Q \) and \( R \) are usually chosen to be diagonal matrices whose entries "penalize" the output/input variables:

ie. Suppose \( y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) and \( u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \)

and \( u_3 \) is a very "expensive" input.

A possible choice of \( Q \) : \( R \) may be:

\[
Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}
\]
It turns out that the solution to the optimal control problem is very interesting:

The optimal control is generated by a specific stabilizing state feedback given by:

\[
F = R^{-1}B^TP
\]

in which \(P\) is the positive definite solution to the Riccati equation

\[
PA + A^TP - PB(R^{-1}B^TP + C^TQC) = 0
\]

Consequently, the optimal feedback loop may be drawn as:

![Feedback Loop Diagram]

\(F = R^{-1}B^TP\)

\[ I \dot{\theta} = u \]

\( u \cdot \text{control torque} \)

let \( x_1 = \theta \); \( x_2 = \dot{\theta} \)

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ \dot{x} = Ax + Bu, \quad \text{assume} \ y = x \]

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

minimize \( J = \int_{0}^{\infty} (x^TQx + u^TRu) \) \( dt \)

choice of \( Q \)?

choice of \( R \)?
EXAMPLE (cont'.)

MATLAB's LQR.m function solves Riccati eqn.

% LQR example
% call_satellite.m

global A;
global B;
global K;

A = [ 0 1; 0 0 ];
B = [ 0; 1 ];
Q = [ 1 0; 0 0 ];
R = 0.1;

[K,P,E] = lqr(A,B,Q,R);

% LQR example
% satellite.m

function [xdot] = satellite(t,x)

global A;
global B;
global K;

xdot = (A-B*K)*x;

x0 = [10 10];
t0 = 0; tf = 20;
[T,x]=ode23('satellite', [t0,tf], x0);

plot(T, x(:,1),T, x(:,2), '--');
As we will now demonstrate, this result is easily proved by "completing the square."

Prove: The optimal control $u^*$ which minimizes

$$J = \int_0^\infty (y^T Q y + u^T R u) \, dt \quad (1)$$

is given by

$$u^* = -R^{-1}B^TPx,$$

where $P$ is the PD solution to the Riccati equation, and $x(t)$ solves

$$\dot{x} = Ax + Bu$$

$$y = Cx.$$

Proof: It follows from standard calculus that

$$\frac{d}{dt} (x^T P x) = \dot{x}^T P x + x^T P \dot{x}$$

for arbitrary matrix $P$ ($n \times n$).
Further,
\[ \int_0^\infty \frac{dx}{dt} (x^T P x) = (x^T P x)(\infty) - (x^T P x)(0) \]  \hspace{1cm} (2)

Combining (1) and (2):
\[ J - (x^T P x)(0) = - (x^T P x)(\infty) \]
\[ + \int_0^\infty (y^T Q y + u^T R u + \frac{d}{dt} (x^T P x)) dt \]

Thus,
\[ J - (x^T P x)(0) = - (x^T P x)(\infty) \]
\[ + \int_0^\infty (y^T Q y + u^T R u + x^T P x + x^T P x) dt \]
\[ = - (x^T P x)(\infty) \]
\[ + \int_0^\infty (y^T Q y + u^T R u + (x^T A^T + u^T B^T) P x + x^T P (A x + B u)) dt \]
\[ = - (x^T P x)(\infty) \]
\[ + \int_0^\infty (x^T (C^T Q C + A^T P + P A) x + u^T R u + u^T B^T P x + x^T P B u) dt \]

By our assumption, \( P \) is the positive definite solution to the Riccati equation:
\[ P A + A^T P - P B R^{-1} B^T P + C^T Q C = 0 \]
Thus
\[ C^TQC + A^T P + PA = PBR^{-1}BTP \]
which we can substitute back in the \( J \) equation; and grouping terms:
\[ \therefore J = (x^T P x)(0) = -(x^T P x)(\infty) + \cdots \]
\[ + \int_0^\infty (u + R^{-1}BTPx)^T R(x + R^{-1}BTPx) \, dt \]

Now, if the system is closed loop stable, then \( (x^T P x)(\infty) = 0 \).
If \( u = -R^{-1}BTPx \), then \( J \) is minimized and in fact
\[ J = (x^T P x)(0) \]

Remarks

1. \( J = x^T(0)P x(0) \) means that the cost depends on \( x(0) \) and \( P \) only.
2. The Riccati equation
\[ A^T P + PA - PB R^{-1}BTP + C^TQC = 0 \]
has many solutions. However there is
only one positive semi-definite solution

\[ P = P^T \geq 0 \]

(3) It can be proven that this solution gives a closed loop matrix of

\[ A - B R^{-1} B^T P \]

which is stable.

(4). But the most remarkable thing is that, out of all possible configurations that one could have for the control input, the optimal control is

linear state feedback!