GOALS OF THIS LECTURE:

- eigenvalue placement by state feedback for MIMO systems

REFS: Callier & Desoer, Chapter 10.
Eigenvalue Placement by State Feedback

MIMO Systems (Multi Input Multi Output)

now $\dot{x} = Ax + Bu$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

let $b \in \mathbb{R}^n$. In general, $(A, b)$ is not c.c.

However, we have the following:

Lemma (Heymann)

let $(A, B)$ be c.c. and $b \in \mathbb{R}^n$. Then there is $F \in \mathbb{R}^{n \times n}$ such that $(A + BF, b)$ is c.c.

Proof. By construction. Define:

$z_1 = b$

$z_{i+1} = Az_i + Bv_i$

where $v_i$ is chosen so that $z_{i+1}$ is independent of $z_1, z_2, z_3, \ldots, z_i$

Claim. The induction terminates at $i = n$.

Proof of claim: If not, $z_i$ such that $\forall v_i$

$Az_i + Bv_i \not\in \text{span} \{z_1, \ldots, z_i\}$

dim $M = i < n$.

In particular, $v_i = 0 \Rightarrow Az_i \in M$

Now since $Az_i + Bv_i \in M \forall v_i \Rightarrow R(B) \subseteq M$. 


and since

\[ z_2 = Az_1 + Bv_1, \]
\[ z_3 = Az_2 + Bv_2. \]

\[ \vdots \Rightarrow Az_1, Az_2, \ldots, Az_{i-1} \in M. \]

Thus \( M \) is an \( A \)-invariant subspace of dimension \( n \) containing \( R(B) \), which contradicts the complete controllability of \((A,B)\). \( \blacksquare \)

Now, going back to the proof of the Lemma:

Define

\[ Fz_1 = v_1, \]
\[ Fz_2 = v_2, \]
\[ \vdots \]
\[ Fz_{n-1} = v_{n-1}, \]
\[ Fz_n = \text{arbitrary}. \]

\[ \text{since } \{z_1, z_2, \ldots, z_n\} \text{ are linearly independent, } F \text{ is well-defined.} \]

Now \( z_1 = b \)

\[ z_2 = Az_1 + Bv_1 = Az_1 + B(Fz_1) = (A+Bf)b \]
\[ z_3 = Az_2 + Bv_2 = (A+Bf)z_2 = (A+Bf)^2 b \]
\[ \vdots \]
\[ z_n = (A+Bf)^{n-1} b. \]

\[ \Rightarrow (A+Bf, b) \text{ is c.c. } \]
Now, given $A, B$ and any monic polynomial $\hat{p}(s) \in R[s]$ with real coefficients, choose $b = Bv \in R(B)$. Then $F_1 \in R^{n \times n_i}$ such that $(A+BF_1, b)$ is c.c.

Now choose

$$f_2^T = \begin{bmatrix} b & (A+BF_1)b & \cdots & (A+BF_1)^{n-1}b \end{bmatrix} \hat{p}(A+BF_1)$$

Then $\lambda (A+BF_1 + bf_2^T) = \text{roots}(\hat{p}(s))$

Thus $F = F_1 + v f_2^T$ places the closed loop eigenvalues at $\hat{p}(s)$.

On the other hand, if $(A, B)$ is not c.c. then $\lambda (A+BF) \forall F$

includes all uncontrollable (or hidden) modes.

ie $\lambda \in \Delta(A)$ with rank $[A - \lambda I \ B] < n$ then $\lambda \in \Delta(A+BF)$

Proof

Let $v^T [A - \lambda I \ B] = \Theta_{n+n_i}^T$

Then $\lambda v^T = v^T A$ and $v^T B = \Theta_{n_i}$

Thus $v^T (A+BF) = \lambda v^T \forall F \in R^{n_i \times n}$

$\therefore \lambda \in \Delta(A+BF)$
Theorem (Stabilizability)

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$. Then $\exists F \in \mathbb{R}^{n \times n}$ such that $\Delta (A + BF) < C$. $\iff \text{rank } [\lambda I - A] = n \forall \lambda \in \mathbb{C}^+.$

Proof

($\Rightarrow$) By contradiction. If $\exists \lambda \in \mathbb{C}^+$ such that $\text{rank } [\lambda I - A] < n$ then $\Delta (A + BF) \geq 1 \forall F \in \mathbb{R}^{n \times n}$ and so $A + BF$ cannot have eigenvalues in $C^0$.

($\Leftarrow$) Since $\text{rank } [\lambda I - A] = n \forall \lambda \in \mathbb{C}^+$, $(A, B)$ can be transformed into the form:

$$
\tilde{A} = \begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} \\
0 & \tilde{A}_{22}
\end{bmatrix}, \quad
\tilde{B} = \begin{bmatrix}
\tilde{B}_1 \\
0
\end{bmatrix}, \quad \text{with } \Delta (\tilde{A}_{22}) < C.
$$

Since $(\tilde{A}_{11}, \tilde{B}_1)$ is cc, $\exists \tilde{F}_1 \in \mathbb{R}^{n \times n}$ such that $\Delta (\tilde{A}_{11} + \tilde{B}_1 \tilde{F}_1) < C$. Choose $\tilde{F} = [\tilde{F}_1 \mid \tilde{F}_2] \in \mathbb{R}^{n \times n}$ where $\tilde{F}_2$ is arbitrary.

Then $\tilde{A} + \tilde{B} \tilde{F} =
\begin{bmatrix}
\tilde{A}_{11} + \tilde{B}_1 \tilde{F}_1 & \tilde{B}_1 \tilde{F}_2 \\
0 & \tilde{A}_{22}
\end{bmatrix}$

and $\Delta (\tilde{A} + \tilde{B} \tilde{F}) = \Delta (\tilde{A}_{11} + \tilde{B}_1 \tilde{F}_1) \cup \Delta (\tilde{A}_{22}) < C$. 