GOALS OF THIS LECTURE:

- observer design
- observer in feedback configuration
- note on eigenvalue placement for SISO and MIMO systems
- separation principle
- reduced order observers
- examples.

REFS: Callier and Desoer, Chapter 10.
Estimator Design in State Space

Recall our STATE FEEDBACK topology:

\[
\overline{N_r} = R + u \\
\]

\[
\begin{array}{cccccc}
& & B & & & \\
& & \oplus & & \int & \\
& & & \circlearrowleft & + & \Delta X \\
& & A & & & \\
& & & F & & \\
\end{array}
\]

\[
\begin{bmatrix}
\chi_1 \\
\chi_2 \\
\end{bmatrix}
\]

\[C \]

\[Y\]

- However, often the state vector is inaccessible for direct measurement.
- Techniques have therefore been developed to provide estimates of inaccessible states.
- An observer is a signal reconstruction device which provides an estimate of inaccessible states.

\[
\overline{N_r} = R + u \\
\]

\[
\begin{array}{cccc}
& & & \text{PLANT} \\
& & (A,B,C) & & \\
& & \oplus & & \Delta \hat{Z} \\
& & F & & \\
\end{array}
\]

\[\hat{Z}\]

\[\text{Observer}\]
Suppose a plant whose dynamical behavior is governed by the state space equations

\[ \dot{x} = Ax + Bu \]
\[ y =Cx \]

has both its input \( u \) and output \( y \) connected to the system below:

If we denote the state of this system as \( \hat{z} \), then an inspection of the diagram shows that its state space equation is:

\[ \dot{\hat{z}} = A\hat{z} + T(y - C\hat{z}) + Bu \]
\[ = (A - TC)\hat{z} + Bu + Ty. \]

Therefore, the dynamical behavior of the difference between the state of the two systems is given by:
\[
(\hat{z} - x) = (A-TC)\hat{z} + Bu - B\mu - AX + T(CX) \\
= (A-TC)(\hat{z} - x)
\]

It therefore follows that if we can choose the feedback matrix \( T \) to be such that the system matrix \((A-TC)\) has negative real parts, then 
\[
\hat{z} \to x \quad \text{as} \quad t \to \infty \quad \text{["asymptotic estimate"]}
\]

irrespective of the plant input \( u \).
Example: Inverted pendulum with Disturbance

\[
\begin{align*}
    x_1 &= \theta \\
    x_2 &= \dot{\theta} \\
    x_3 &= d
\end{align*}
\]

\[
\begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= \Omega^2 x_1 - \alpha x_2 + x_3 + u \\
    \dot{x}_3 &= 0
\end{align*}
\]

\[
    y = x_1
\]

\[
    \therefore A = \begin{bmatrix}
        0 & 1 & 0 \\
        \Omega^2 & -\alpha & 1 \\
        0 & 0 & 0
    \end{bmatrix}, \quad B = \begin{bmatrix}
        0 \\
        1 \\
        0
    \end{bmatrix}, \quad C = \begin{bmatrix}
        1 & 0 & 0
    \end{bmatrix}
\]

The observer dynamics are therefore given by:

\[
\dot{\hat{x}} = (A - TC) \hat{x} + Bu + Ty
\]

\[
= \begin{bmatrix}
    -T_1 & 1 & 0 \\
    \Omega^2 - T_2 & -\alpha & 1 \\
    -T_3 & 0 & 0
\end{bmatrix} \hat{x} + \begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix} u + \begin{bmatrix}
    T_1 \\
    T_2 \\
    T_3
\end{bmatrix} y
\]
We have seen how we may use the plant input and output to generate an asymptotic estimate of the state vector.
Now suppose we actually feed back the state estimate to the plant input as shown above.

The equations describing the entire system (plant + observer + controller) are:

\[ \dot{\hat{z}} = (A - BF - TC) \hat{z} + TCX \]
\[ \dot{x} = AX - BF \hat{z} \]
Now since the error between the state estimate and the actual state is \( e(t) = \hat{e}(t) - x(t) \),

\[
\begin{align*}
\dot{e} &= (A - TC) \hat{e} - (A - TC)x \\
&= (A - TC) e \\
&= (A - TC) e \\
&= (A - BF) x - BFE
\end{align*}
\]

Write the state equation in terms of state and error:

\[
\begin{align*}
\dot{x} &= Ax - BF \hat{e} \\
&= Ax - BF(e + x) \\
&= (A - BF)x - BFE
\end{align*}
\]

Combining (*) and (**):

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} = 
\begin{bmatrix}
A - BF & -BF \\
0 & A - TC
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\]

and since the eigenvalues of a block diagonal matrix are the combined eigenvalues of the diagonal blocks, we have that the eigenvalues of the composite system are those of \((A - BF)\) and \((A - TC)\).

... "Separation Principle"...
The Separation Theorem

The problem of arbitrarily assigning the closed loop poles of a system using feedback can be separated into two parts:

(i) designing an observer to provide a set of asymptotically accurate state estimates;

(ii) designing a pole-assigning state feedback matrix as though the true states were available for direct measurement.
Algorithm for Pole Placement in Observer Design

Calculate $T$ so that $(A - Tc)$ has desired eigenvalues:

Recall that the set of eigenvalues of any square matrix and of its transpose are the same. Therefore the eigenvalues of $(A - Tc)$ are the same as those of $(A^T - c^TT^T)$.

Which is exactly the same problem we solved in the pole placement algorithm of lectures 17 & 18 with

- $A$ replaced by $A^T$
- $B$ replaced by $c^T$
- $F$ replaced by $T^T$.

So just use the same algorithm to solve for $T$. We say that the standard observer configuration is the dual of the state feedback configuration.
example

\[
\dot{X} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \\
Y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} X
\]

1. Design an observer with poles at 
   \(-4\) and \(-4 \pm j2\)

2. Design a state feedback so that the closed loop poles are located at 
   \(-2\) and \(-2 \pm j2\).

Solution

1. Observer design:

\[
(A - TC) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & 0 & -T_1 \\ 1 & 0 & -T_2 \\ 0 & 1 & -1-T_3 \end{bmatrix}
\]

Characteristic equation:

\[
\det(\lambda I - (A - TC)) = 0
\]
\[
\iffalse
\lambda^3 + (1 + T_3)\lambda^2 + T_2\lambda + T_1 = 0
\fi
\]

**Desired characteristic polynomial for observer:**

\[
(\lambda + 4)(\lambda + 4 + 2j)(\lambda + 4 - 2j) = 0
\iffalse
\Rightarrow \lambda^3 + 13\lambda^2 + 60\lambda + 100 = 0
\fi
\]

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3
\end{bmatrix}
= \begin{bmatrix}
100 \\
60 \\
12
\end{bmatrix}
\]

2. **State feedback design:**

\[
(A - BF) = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & -1
\end{bmatrix} - \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\iffalse
= \begin{bmatrix}
-F_1 & -F_2 & -F_3 \\
1 & 0 & 0 \\
0 & 1 & -1
\end{bmatrix}
\fi
\]

**Characteristic equation:**

\[
\det (\lambda I - (A - BF)) = 0
\iffalse
\Rightarrow \lambda^3 + \lambda^2(-1 - F_1) + \lambda(-F_1 - F_2) - F_2 - F_3 = 0
\fi
\]

**Desired:**

\[
(\lambda + 2)(\lambda^2 + 4\lambda + 8) = 0
\iffalse
(\lambda^3 + 6\lambda^2 + 16\lambda + 16) = 0
\fi
\]

\[
F = \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix} = \begin{bmatrix}
-5 \\
-11 \\
-5
\end{bmatrix}
\]
Where should the closed loop poles and estimator poles be placed?

**Closed loop poles:**

\[ u = -Fx \]

- Control input is proportional to gain \( F \). The larger the gain, the larger the control input.

- The less the poles are moved from open loop to closed loop, the smaller the gain matrix.

**Estimator poles:**

- Chosen (usually) faster than controller poles – gives a faster decay of estimator errors compared with desired dynamics.

- Usually a bad idea to move estimator poles too far to the left, since this increases the bandwidth of the estimator, causing more sensor noise to pass on to the control actuator.
Reduced Order Observer Design

Now we know that

\[ y = Cx \]

\[ \begin{bmatrix} 1 \\ \vdots \\ y \\ C \end{bmatrix} \begin{bmatrix} \hat{x} \\ \vdots \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} \hat{x} \\ \vdots \end{bmatrix} \]

\[ y = C \begin{bmatrix} \hat{x} \\ \vdots \end{bmatrix} \]

\[ C \] is \( p \times n \)
\[ p \leq n \text{ and } \text{rank} \ C \leq p \]

otherwise we wouldn't need an observer

constructed full order observers (ie. of dimension \( n \), same as plant). But this may be needlessly complicated, especially if several of the states are directly measured in \( y \).

Here, let us consider the design of a reduced order observer:

\[ C \in \mathbb{R}^{p \times n} \]
\[ T \in \mathbb{R}^{(n-p) \times n} \]

where \( \det [C] \neq 0 \)
Assume (without loss of generality) that $C$ has full row rank:

\[ \begin{bmatrix} T^1 \\ C \end{bmatrix} \]

and consider

\[ \dot{z} = Mz + Nu + Ly \]

for some matrices $M, N, L$.

Now, as $t \to \infty$, we'd like $z \to T x$

\[ \frac{d}{dt} (z - T x) = Mz + Nu + Ly - T (Ax + Bu) \]

\[ = M(z - T x) + MT x - Ta x + LC x \]

\[ = M(z - T x) + (MT - Ta + LC) x \]

now, we'd like $z - T x \to 0$

\[ \therefore \text{let } N - TB = 0 \]

\[ MT - Ta + LC = 0 \]

$M$ be asymptotically stable (all eigenvalues in open left half plane)

Thus, possible design steps could be:

1. Choose stable $M$ (i.e., diagonal - easy)
2. Choose $L$
3. Solve for $T$ in $MT - Ta + LC = 0$
4. $N = TB$ solve for $N$

5. check $\det \begin{bmatrix} T \\ C \end{bmatrix} \neq 0$.

While this design procedure may be ok for simple systems - it is not recommended in general, because it gives you no control over the matrix $\begin{bmatrix} T \\ C \end{bmatrix}$. If $\begin{bmatrix} T \\ C \end{bmatrix}$ is close to being singular - its inverse would result in a huge gain in the backward loop.

Consider instead the following Reduced order observer design procedure:

1. Transform the plant $\dot{x} = Ax + Bu$ to a new set of coordinates $\bar{x}$, using similarity transform $S, S^{-1}$: $x = S \bar{x}$

   such that $\bar{C} = CS = [C_1, 0]$

   where $C_1$ is a $p \times p$ non-singular matrix.

   [ie. to find $S$, do column operations on $C$ to put it into column echelon form - use column pivoting]
\[ \begin{align*}
\bar{C} &= CS = \begin{bmatrix} C_1 & 0 \end{bmatrix} \quad C_1 \in \mathbb{R}^{p \times p} \\
\bar{A} &= S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad A_{11} \in \mathbb{R}^{p \times p} \\
\bar{B} &= S^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad B_1 \in \mathbb{R}^{p \times m}
\end{align*} \]

2. Design an observer for \( \bar{x} \) (i.e., construct state estimate \( \hat{z} \) such that \( z \rightarrow \bar{x} \)):

Use
\[
\begin{bmatrix} \bar{C} \\ T \end{bmatrix}^{-1} = \begin{bmatrix} C_1 & 0 \\ -T_1 & I \end{bmatrix}^{-1} \quad \text{which is non-singular by construction.}
\]

And now solve for \( T_1 \):

\[ \begin{align*}
0 &= MT - T\bar{A} + \bar{L}\bar{C} \\
(*) &= M \begin{bmatrix} -T_1 & I \end{bmatrix} - \begin{bmatrix} -T_1 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \bar{L} \begin{bmatrix} C_1 & 0 \end{bmatrix} \\
\text{i.e.} \quad M + T_1 A_{12} - A_{22} &= 0 \\
\therefore \quad M &= A_{22} - T_1 A_{12} \quad \text{which is} \ldots \]
... a standard pole placement problem!

ie. recall \((A - BF)^T = A^T - FTB^T\)

\[ A_{22} - T_1 A_{12} = M \]

\[ \uparrow \]

want \(M\) to be stable.

.: Given \((A_{22}, A_{12})\), and desired eigenvalues for \(M\), the design of \(T_1\) simply follows regular pole placement design!

It is easy to show that \((A, C)\) observable implies that \((A_{22}, A_{12})\) is controllable.

3. \(-\) From (*):

\[-MT_1 - [-T_1 \ I] \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + LC_1 = 0\]

\[\Rightarrow L = (MT_1 + [-T_1 \ I] \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}) C_1^{-1}\]

4. \(N = T \bar{B} = [-T_1 \ I] \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}\)
Example Design a reduced order observer for \( \dot{x} = Ax + Bu \)
\( y = Cx \)
with \( A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \)
\( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)
\( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \)

Here, the order of the observer is 1:

\[ n = 2 \]
\[ p = 1 \]

1. Note, \( C \) is already in column echelon form:
\[ \therefore C_1 = 1 \]

2. Let \( T = [-t_1, 1] \)
\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \]

Choose \( M = -1 \)
\[ \therefore M = A_{22} - t_1 A_{12} = 0 - t_1 \cdot 1 = -t_1 \]
\[ \Rightarrow t_1 = 1 \]

3. \[ L = (MT_1 - t_1 A_{11} + A_{21})C_1^{-1} \]
\[ = (-t_1^2 - t_1 - t_1 - (2)) \cdot 1^{-1} \]
\[ = -2 - t_1^2 = -3 \]

4. \[ N = TB = [-t_1, 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \]