Goals of this lecture:

- introduce linear maps.
  - range space
  - nullspace
- introduce matrix representation of a linear map.
- matrix representation under change of basis.
  - similarity transform.
- matrix manipulations
  - elementary row, col. operations
- Sylvester's Inequality.

References

Appendix A C D
S A.4, A.5
Linear Maps

Let \((V, F)\) and \((W, F)\) be linear spaces over the same field \(F\). Let \(A\) be a map from \(V\) to \(W\):

\[ A : V \rightarrow W \]

\[ \text{st } A(v) = w \]

\[ \forall v \in V, w \in W \]

Then \(A\) is said to be a linear map (equiv. linear operator) iff

\[ A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A(v_1) + \alpha_2 A(v_2) \]

\[ \forall \alpha_1, \alpha_2 \in F \]

\[ \forall v_1, v_2 \in V \]

* "map" notation ... "\(A\) operates on an element \(\alpha_1 v_1 + \alpha_2 v_2\) in \(V\)." We will show that this "operation" is equivalent to pre-multiplication of \(\alpha_1 v_1 + \alpha_2 v_2\) by a matrix, if \(A\) is linear.

Example: Consider the following mapping on the set of polynomials of degree 2:

\[ A : as^2 + bs + c \rightarrow cs^2 + bs + a \]

is this a linear map?
Solution: let \( v_1 = a_1 s^2 + b_1 s + c_1 \),
\( v_2 = a_2 s^2 + b_2 s + c_2 \)

\[
A(x, v_1 + x_2 v_2)
= \frac{1}{2} \left( x a_1 s^2 + x b_1 s + x_1 c_1 + x_2 a_2 s^2 + x_2 b_2 s + x_2 c_2 \right)
= \left( a_1 c_1 + x_2 c_2 \right) s^2 + \left( a_1 b_1 + x_2 b_2 \right) s + \left( x_1 a_1 + x_2 a_2 \right)
= n(x, v_1) + n(x_2, v_2)
\]

\[
:: A \text{ is a linear map}
\]

*exercise how about
\[
A: a s^2 + b s + c \rightarrow \int (bt + a) \, dt
\]

*exercise how about
\[
A: v(t) \rightarrow \int v(t) \, dt + k
\]
for \( v(\cdot) \in C([0,1], \mathbb{R}) \).

*exercise and \( A: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)
\[
A(v) := Av \text{ where } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 7 & 0 & 16 \end{bmatrix}
\]

* note what happens to the zero vector under this (or any) linear map!
Def. Given a linear map \( A : U \to V \), define the **range space** of \( A \) to be the subspace
\[ R(A) := \{ v \in V \mid \exists u \in U : A(u) = v \} \]
and define the **nullspace** of \( A \) to be the subspace
\[ N(A) := \{ u \in U \mid A(u) = 0 \} \]
\( R(A) \) is also called the **image** of \( A \); \( N(A) \) is also called the **kernel** of \( A \).

*Exercise.* Prove that \( R(A) \) and \( N(A) \) are linear subspaces.

Theorem (Range and Null Space of Linear Operators)
Consider \( A : U \to V \) with \((U,F),(V,F)\) linear spaces. Let \( b \in V \). Then:

(a) \( A(u) = b \) has at least one solution \( \iff b \in R(A) \)

(b) If \( b \in R(A) \) then
   (i) \( A(u) = b \) has a unique solution \( \iff N(A) = \{0\} \)
   (ii) Let \( x_0 \) be such that \( A(x_0) = b \). Then \( A(x) = b \iff x - x_0 \in N(A) \).

Proof * exercise.
Exercise let \( A \) be a linear map from 
\((U, F) \) to \((V, F) \) with \( \dim U = n \) and 
\( \dim V = m \). Then show that:
\[
\dim \ker(A) + \dim \text{Im}(A) = n.
\]

**Matrix Representation**

Any linear map between finite dimensional 
linear spaces can be represented as matrix 
multiplication.

let \( A : U \to V \) be a linear map from \((U, F) \) 
to \((V, F) \) where \( \dim U = n \) and \( \dim V = m \). 
let \( \xi_j \; j = 1 \) be a basis for \( U \) and let 
\( \xi_j \; j = 1 \) be a basis for \( V \).

Thus, for any \( x \in U, \exists! \xi = \{x_j \xi_j \} \in F^n \) 
such that \( x = \sum_{j=1}^{n} \xi_j u_j \).

By linearity, \( A(x) = A(\xi_j \xi_j u_j) = \xi_j \xi_j A(u_j) \)
now, each \( A(u_j) \in V \), thus each \( A(u_j) \) has a 
unique representation in terms of the \( \xi_j \; j = 1 \):
\[
A(u_j) = \sum_{i=1}^{m} a_{ij} \xi_i \quad V_j \in \{1, \ldots, m\} \quad j^{th} \text{col} \ A.
\]

ie. \( A(u_1) = \sum_{i=1}^{m} a_{1i} \xi_i \), \( A(u_n) = \sum_{i=1}^{m} a_{ni} \xi_i \) 
thus \( a_{ij} \xi_i \) is the representation of \( A(u_j) \) in 
terms of \( \xi, \xi_2 \ldots \xi_m \).
\[ A(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \xi_j \nu_i \]
\[ = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} \xi_j \right) \nu_i \]
\[ = \sum_{i=1}^{m} \eta_i \nu_i \]

Thus, the representation of \( A(x) \) with respect to \( \xi \nu_1, \xi \nu_2 \ldots \xi \nu_m \) is \( \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix} \in F^m \) and the representation of \( x \) with respect to \( \xi \nu_1, \xi \nu_2 \ldots \xi \nu_m \) is \( \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \in F^n \).

By uniqueness of the representation:
\[ \eta_i = \sum_{j=1}^{n} a_{ij} \xi_j \quad \text{in each } i \in \{1, \ldots, m\} \]

ie. \( \eta = A \xi \), \( A \in F^{m \times n} \).

where \( A \) is the matrix representation of the linear operator \( A \) from \( U \) to \( V \).

**Good to remember:** The \( j^{th} \) column of the matrix \( A \) is \( A(\nu_j) \) expressed with respect to \( \xi \nu_1 \ldots \xi \nu_m \).

**Example**

Let \( A : (R^n, R) \rightarrow (R^n, R) \)
\[ A^m = -x_1 A^{m-1} - x_2 A^{m-2} \ldots - x_{n-1} \lambda - x_n I, \quad \lambda \in R \]
\[ b \in R^n \]

Suppose \( (b, A(b), \ldots, A^{m-1}(b)) \) is a basis for \( R^n \).

Show that, with respect to this basis, the vector
b and the linear map A are represented by:
\[
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

\[
A =
\begin{bmatrix}
0 & 0 & \cdots & 0 & -\alpha_n \\
1 & 0 & \cdots & 0 & -\alpha_{n-1} \\
0 & 1 & \cdots & 0 & -\alpha_{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -\alpha_1
\end{bmatrix}
\]

**Solution**

\[
\overline{b} = 1 \cdot b + 0 \cdot A(b) + \cdots + 0 \cdot A^{n-1}(b) =
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

\[
A(b) = 0 \cdot b + 1 \cdot A(b) + \cdots + 0 \cdot A^{n-1}(b)
\]

\[
A(A(b)) = 0 \cdot b + 0 \cdot A(b) + 1 \cdot A^2(b) + \cdots + 0 \cdot A^{n-1}(b)
\]

\[
A(A^{n-1}(b)) = A^n(b) = -\alpha_1 b - \alpha_{n-1} A(b) + \cdots -\alpha_1 A^{n-1}(b)
\]

hence

\[
A =
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

as expected.

Now, consider the maps

\[
\begin{align*}
A &: U \rightarrow V \\
B &: V \rightarrow W
\end{align*}
\]

let \( \xi_j \) \( j = 1, \ldots, n \) be a basis for \((U, F)\)

\[
\xi_j \] \( j = 1, \ldots, m \) \quad \text{and} \quad \xi_j \] \( j = 1, \ldots, p \)

and let \( \xi \in F^n \), \( \eta \in F^m \), and \( \zeta \in F^p \) be the corresponding component vectors.

Thus, with \( A \in F^{m \times n} \) and \( B \in F^{p \times m} \), the matrix representations of the linear maps \( A \) and \( B \) w.r.t. the above bases.
Thus, \( \eta = A \xi \) and \( \xi = B \eta \),
and thus \( \xi = BA \xi := C \xi \)
\( \Rightarrow \) the composition of linear maps
corresponds to matrix multiplication.

Change of Basis

Here, we will study the relationship between
two matrix representations of the same linear map.

Let \((u_j)^n\) and \((\overline{u}_i)^n\) be two bases for \((U, F)\), and
\((v_i)^m\) and \((\overline{v}_i)^m\) for \((V, F)\).

Let \( A \) be the matrix representation of \( A : U \rightarrow V \)
w.r.t. the bases \((u_j)^n\) and \((v_i)^m\);
and let
\( \overline{A} \) be the matrix representation of \( \overline{A} \)
w.r.t. the bases \((\overline{u}_i)^n\) and \((\overline{v}_i)^m\).
Now $\mathbf{u} \in \mathbb{R}^n$, 
\[ \mathbf{x} = \xi_1 \mathbf{u}_1 + \xi_2 \mathbf{u}_2 + \cdots + \xi_n \mathbf{u}_n \]
\[ = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n] \xi \quad \text{where} \quad \xi = \left[\begin{array}{c} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{array}\right] \in \mathbb{R}^n \]

but $\mathbf{x} = \xi_1 \mathbf{u}_1 + \xi_2 \mathbf{u}_2 + \cdots + \xi_n \mathbf{u}_n$
\[ = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n] \xi \quad \text{where} \quad \xi = \left[\begin{array}{c} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{array}\right] \in \mathbb{R}^n \]

\[ \because [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n] \xi = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n] \xi \]
\[ \Rightarrow \xi = \mathbf{P} \xi \quad \text{where} \quad \mathbf{P} = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n]^{-1} [\mathbf{u}_1 \cdots \mathbf{u}_n] \]
and $\mathbf{\bar{u}} = \mathbf{Q} \eta$ 
$\mathbf{Q} = [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_n]^{-1} [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_n]$

now $\eta = \mathbf{A} \xi$

\[ \therefore \mathbf{\bar{u}} = \mathbf{Q} \mathbf{A} \xi = \mathbf{Q} \mathbf{A} \mathbf{P} \xi \]
\[ = \mathbf{A} \xi \quad \text{where} \quad \mathbf{\bar{A}} = \mathbf{Q} \mathbf{A} \mathbf{P} \]

So if $\mathbf{A}$ is the matrix representation of $\mathbf{A}$ wrt $\xi, \xi$, then $\mathbf{\bar{A}} = \mathbf{Q} \mathbf{A} \mathbf{P}$ is the matrix representation of $\mathbf{A}$ wrt $\xi'. \xi'$.

Note: $\mathbf{A}$ and $\mathbf{\bar{A}}$ are said to be equivalent, and $\mathbf{\bar{A}} = \mathbf{Q} \mathbf{A} \mathbf{P}$ is said to be a similarity transformation.

Note: if $\mathbf{U} = \mathbf{V}$ and $(\mathbf{v}_j)' = (\mathbf{u}_j)'$, then
\[ \mathbf{\bar{A}} = \mathbf{P}^{-1} \mathbf{A} \mathbf{P} \]
example: let $A: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. Consider $B = \{b_1, b_2, b_3\} = \{[1,0,0], [0,1,0], [0,0,1]\}$

$C = \{c_1, c_2, c_3\} = \{[1,1,0], [0,1,0], [0,0,1]\}$

Clearly $B$ and $C$ are bases for $\mathbb{R}^3$. Suppose $A$ maps:

$A(b_1) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $A(b_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $A(b_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Write down the matrix representation of $A$ wrt $B$ and then wrt $C$.

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By rank of the matrix $A \in \mathbb{F}^{m \times n}$ (denoted as $\text{rank}(A)$ or $\text{rk}(A)$) we mean $\text{dim}(R(A))$; by nullity of $A \in \mathbb{F}^{m \times n}$ (denoted as $\text{null}(A)$) we mean $\text{dim}(N(A))$.

- row rank; full row rank
- col rank; full col. rank

Sylvester's Inequality

let $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times p}$, then $AB \in \mathbb{F}^{m \times p}$ and

$\text{rk}(A) + \text{rk}(B) - n \leq \text{rk}(AB) \leq \min(\text{rk}(A), \text{rk}(B))$

Proof: * exercise (homework #1).

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Finally, we will be interested in reducing matrix \( A \in F^{m \times n} \) to row or column echelon form, as these are well suited for discussing the construction of a basis for \( R(A) \), \( N(A) \).

**Elementary Row Operations (e.r.o.)**

(i) interchange two rows: \( r_i \leftrightarrow r_j \)
(ii) multiply row \( i \) by a non-zero \( c \in F \): \( r_i \rightarrow c r_i \), \( c \neq 0 \)
(iii) add to row \( i \) another row \( j \): \( r_i \rightarrow r_i + c r_j \)

Ero's are equivalent to premultiplying \( A \) by left elementary matrix \( L \), which is obtained from the identity matrix by performing the desired e.r.o upon it. i.e: \( L \cdot A \).

*Show* \( N(L \cdot A) = N(A) \)

**Elementary column operations (e.c.o.)**

(i) interchange two columns \( \gamma_i \leftrightarrow \gamma_j \)
(ii) multiply col. \( j \) by a non-zero conts: \( \gamma_i \rightarrow c \gamma_i \)
(iii) \( \gamma_i \rightarrow \gamma_i + r \gamma_j \)

--- corresponds to postmultiplying \( A \) by right elementary matrix \( R \), which is obtained from the identity by performing desired e.c.o on it, i.e. \( A \cdot R \).

*Show* \( R(A \cdot R) = R(A) \).