1 Range Space and Null Space of Adjoint Maps

Consider a linear map $A : U \to V$.

**Proposition 1.**

1) $V = R(A) \oplus N(A^*)$ and $R(A)^\perp = N(A^*)$.

2) $U = R(A^*) \oplus N(A)$ and $R(A^*)^\perp = N(A)$.

**Proposition 2** (The “Very Useful Proposition”).

1) $N(AA^*) = N(A^*)$

2) $R(AA^*) = R(A)$

3) $N(A^*A) = N(A)$

4) $R(A^*A) = R(A^*)$

**Problem 1.** Show that $R(A^*A) = R(A^*)$.

2 Definition of Controllability and Observability

Consider a dynamical system $D = (U, \Sigma, Y, s, r)$.

**Definition 3.** The system $D$ is *completely controllable* (or just “controllable”) on $[t_0, t_1]$ iff any $(x_0, t_0)$ can be transferred to any $(x_1, t_1)$ by some $u_{[t_0, t_1]} \in U$.

**Remark 1.** In other words, $\forall x_0 \in \Sigma$ the map $s(t_1, t_0, x_0, u_{[t_0, t_1]}): U \to \Sigma$ is surjective.
Definition 4. The system $D$ is completely observable on $[t_0, t_1]$ if, for all $u_{[t_0, t_1]} \in \mathcal{U}$ and for all $y_{[t_0, t_1]} \in \mathcal{Y}$, $x_0$ at $t_0$ is uniquely determined.

Remark 2. In other words, $\forall y_{[t_0, t_1]} \in \mathcal{Y}$ the map $\rho(t_1, t_0, x_0, u_{[t_0, t_1]}): \Sigma \to \mathcal{Y}$ is injective.

Consider. These definitions are intuitive, but hard to solve. Can we find a more concrete test for complete controllability (c.c.) and complete observability (c.o.)?

3 Controllability of LTV systems

Consider the Linear Time-Varying system

$$
\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (1)
$$

$$
y(t) = C(t)x(t) + D(t)u(t) \quad (2)
$$

We know that

$$
s(t_1, t_0, x_0, u_{[t_0, t_1]}) = x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \quad (3)
$$

From lecture we learned that finding surjectivity of eq (3) is equivalent to finding surjectivity of:

$$
x_1 - \Phi(t_1, t_0)x_0 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau =: \mathcal{L}_c(u_{[t_0, t_1]}) \quad (4)
$$

where $\mathcal{L}_c: \mathcal{U}_{[t_0, t_1]} \to \mathbb{R}^n$. Proving surjectivity of $\mathcal{L}_c$ is equivalent to proving $\mathcal{R}(\mathcal{L}_c) = \mathbb{R}^n$.

Consider. Now we know that $\mathcal{R}(\mathcal{L}_c) = \mathbb{R}^n \iff$ surjectivity of $s(t_1, t_0, x_0, u_{[t_0, t_1]}) \iff$ c.c.

This is a bit easier to solve for, but we can make it even easier.

4 Controllability Grammian

1. From above, complete controllability $\iff \mathcal{R}(\mathcal{L}_c) = \mathbb{R}^n$

2. We know $\mathcal{R}(AA^*) = \mathcal{R}(A)$, so $\mathcal{R}(\mathcal{L}_c\mathcal{L}_c^*) = \mathcal{R}(\mathcal{L}_c)$.

3. From lecture we derived that $\mathcal{L}_c^* = B^*(t)\Phi^*(t, t_0)$

4. We can define a map $W_{c_{[t_0, t_1]}} := \mathcal{L}_c\mathcal{L}_c^*$, where $W_{c_{[t_0, t_1]}: \mathbb{R}^n \to \mathcal{U}_{[t_0, t_1]} \to \mathbb{R}^n$.

5. The domain and codomain of $W_c$ are both in $\mathbb{R}^n$, so we can represent this as a matrix:

$$
\mathcal{L}_c(u_{[t_0, t_1]}) = \int_{t_0}^{t_1} \Phi(\tau, t_0)B(\tau)u(\tau)d\tau
$$

$$
\mathcal{L}_c(\mathcal{L}_c^*) = W_{c_{[t_0, t_1]}} = \int_{t_0}^{t_1} \Phi(\tau, t_0)B(\tau)\mathcal{L}_c^*_{[t_0, t_1]}d\tau
$$

$$
= \int_{t_0}^{t_1} \Phi(\tau, t_0)B(\tau)B^*(\tau)\Phi^*(\tau)d\tau
$$

This results in $W_{c_{[t_0, t_1]}}$ being a square, positive semidefinite (PSD) matrix of size $n \times n$. 
Observability follows very similar logic to controllability. 

\[ y(t_1) = \rho(t_1, t_0, x_0, u_{[t_0, t_1]} = C(t_1)\Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} C(t_1)\Phi(t_1, \tau)B(\tau)u(\tau)d\tau + D(t_1)u(t_1) \]

Let \( L_o : \mathbb{R}^n \rightarrow \mathcal{Y}_{[t_0, t_1]} \) such that 

\[ L_o x_0 = C(\cdot)\Phi(\cdot, t_0)x_0. \]

Then,

\[ y(t) = L_o x_0(t) + \int_{t_0}^{t_1} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t). \]

1. An LTV system is observable on \([t_0, t_1]\) if and only if \( N(L_o) = \{0_n\}. \)

2. We know \( N(A^*A) = N(A) \), so \( N(L_o^*L_o) = N(L_o) \)

3. From lecture we derived that \( L_o^*(y_{[t_0, t_1]}) = \int_{t_0}^{t_1} \Phi^*(\tau, t_0)C^*(\tau)y(\tau)d\tau \)

4. We can define a map \( W_o_{[t_0, t_1]} := L_o^*L_o \), where \( W_o_{[t_0, t_1]} : \mathbb{R}^n \rightarrow \mathcal{U}_{[t_0, t_1]} \rightarrow \mathbb{R}^n \)

5. The domain and codomain of \( W_o \) are both in \( \mathbb{R}^n \), so we can represent this as a matrix:

\[ L_o^*(y_{[t_0, t_1]}) = \int_{t_0}^{t_1} \Phi^*(\tau, t_0)C^*(\tau)y(\tau)d\tau \]
\[ L_o^*(L_o) = W_o_{[t_0, t_1]} = \int_{t_0}^{t_1} \Phi^*(\tau, t_0)C^*(\tau)L_od\tau \]

This results in \( W_o_{[t_0, t_1]} \) being a square, positive semidefinite (PSD) matrix of size \( n \times n \).

6. If \( \text{rank}(W_o) = n \), then \( \text{null}(W_o) = 0 \) (rank nullity). Therefore:

\[ \text{rank}(W_o) = n \iff N(W_o) = N(L_o^*L_o) = N(L_o) = 0 \iff c.o. \]

So we just have to check the rank / positive definiteness of \( W_o \)!

**Summary**

- Controllable on \([t_0, t_1] \iff R(L_o) = \mathbb{R}^n \iff R(L_o^*L_o) = \mathbb{R}^n \iff \text{rank}(W_o) = n \iff x^TW_ox > 0 \)

- Observable on \([t_0, t_1] \iff N(L_o) = \{0_n\} \iff N(L_o^*L_o) = \{0_n\} \iff \text{rank}(W_o) = n \iff x^TW_ox > 0 \)
6 Controllability and observability of LTI systems

Consider the Linear Time-Invariant system \( \dot{x} = Ax + Bu \), \( y = Cx + Du \)

**Theorem 5.**
The system is completely controllable on \([0, \Delta]\) for some \( \Delta > 0 \)
\[\iff \text{rank} \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = n \iff \text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} = n \text{ for all } s \in \mathbb{C} \]

**Theorem 6.**
The system is completely observable on \([0, \Delta]\) for some \( \Delta > 0 \)
\[\iff \text{rank} \begin{bmatrix} C & CA & \ldots & CA^{n-1} \end{bmatrix} = n \iff \text{rank} \begin{bmatrix} sI & -A \end{bmatrix} = n \text{ for all } s \in \mathbb{C} \]

**Problem 2.** Consider the controllability and observability Grammians \( W_c, W_o \) of a linear time invariant system \((A, B, C)\) over the time period \([0, T]\). Determine what happens to them under similarity transformations of the state space. That is, determine the controllability and observability Grammians of \((TAT^{-1}, TB, CT^{-1})\), where \(T\) is a nonsingular matrix. Prove that the eigenvalues of the product \( W_c W_o \) are constant under similarity transformations.

**Problem 3.** For each of the following, provide either a proof or a counterexample:

(a) Suppose \((A, B)\) is controllable. Is the system \((A^2, B)\) controllable?

(b) Suppose \((A^2, B)\) is controllable. Is the system \((A, B)\) controllable?
7 Controller and Observer Design

Problem 4 (Output feedback design). Consider the linear system defined by

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]
\[ y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \]

(a) Is the system controllable? Is it observable?

(b) Can the closed loop poles of the system be placed at \( \lambda_1 = -2, \lambda_2 = -2 \) using output feedback alone?

Now consider the same plant with an additional state measurement state such that

\[ y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \]

(c) Is the system still controllable and observable?

(d) Can the closed loop poles of the system be placed at \( \lambda_1 = -2 \) and \( \lambda_2 = -2 \)?

(e) Explain how the closed loop poles of the system could be placed at \( \lambda_1 = -2 \) and \( \lambda_2 = -2 \) using only a single sensor, i.e., the output is one-dimensional.
Problem 5 (Observer design). Consider the linear system defined by

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{align*}
\]

Can you design an observer for this system which has three poles at $-2$?
Problem 6. An approximate linear model of the longitudinal dynamics of certain aircraft, for a particular set of conditions, has the linearized state and control vectors:

\[
x = \begin{bmatrix} v \\ \alpha \\ \theta \\ q \end{bmatrix}, \quad u = \begin{bmatrix} \delta \\ \mu \end{bmatrix}
\]

where \( v \) represents change in forward velocity, \( \alpha \) the change in angle of attack, \( \theta \) the change in pitch angle, and \( q \) the change in pitch rate. The two inputs are \( \delta \), the deflection of the elevators, and \( \mu \), the throttle position. The state space equation for this model is \( \dot{x} = Ax + Bu \) where

\[
A = \begin{bmatrix}
-0.045 & 0.036 & -32 & -2 \\
-0.4 & -3 & -0.3 & 250 \\
0 & 0 & 0 & 1 \\
0.002 & -0.04 & 0.001 & -3.2
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0.1 \\
-30 & 0 \\
0 & 0 \\
-10 & 0
\end{bmatrix}
\]

1. Suppose a malfunction prevents manipulation of the input \( \delta \). Is it possible to completely control the aircraft using only \( \mu \)? What if only \( \delta \) is available?

2. If you had your choice of only one of the following sensors, which would you use? Would it make a difference? Explain.
   - A rate gyro which measures the pitch rate \( q \).
   - A pitch indicator which measures \( \theta \).
Consider the dynamical model
\[
\begin{align*}
\ddot{p}(t) &= -\frac{m}{M}g\theta(t) + \frac{1}{M}u(t) \\
\ddot{\theta}(t) &= \frac{M + m}{M\ell}g\theta(t) - \frac{1}{M\ell}u(t)
\end{align*}
\]
where $M, m, \ell, g$ are positive constants. This model describes the linearized equations of motion of an inverted pendulum where $p(t)$ is the position of the cart, $\theta(t)$ is the angle of the pendulum, and $u(t)$ is the input force.

a) Is it possible to design an asymptotically stabilizing static controller that uses only $\theta(t)$ measurements for feedback?

b) What if we allow a dynamic controller?

c) What about a dynamic controller that uses only $p(t)$ measurements?

d) How would you answer part c if $m \ll M$?
1. State Equations

a. For the circuit in figure below, write the state equations in the form
\[ \dot{x} = Ax + Bu, \quad y = Cx \text{ using } \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \text{ as the state vector.} \]

b. Is this system controllable and observable?

c. Is this system BIBO stable? Why or why not?

\[ \text{For an unknown } x(t = 0), \text{ given } y(t), \text{ can a } u(t) \text{ be found such that } x(t) \to 0 \text{ as } t \to \infty? \ (\text{Or } x(t_f) = 0 \text{ for some } t_f > 0). \]

\[ \text{For a known } x(t = 0) = x_0, \text{ given } y(t), \text{ can a } u(t) \text{ be found such that } x(t) \to 0 \text{ as } t \to \infty? \]

2. State Equations

For the circuit in figure below, now consider an output \( y_1 = V_1 + V_2 \).

a. Is this system controllable and observable?

b. For an unknown \( x(t = 0) \), given \( y_1(t) \), can a \( u(t) \) be found such that \( x(t) \to 0 \) as \( t \to \infty? \) If so, find such a \( u(t) \).

3. Qualitative behavior

Let \( C_1 = C_2 = 1F, R = 10\Omega \).

a. Sketch the phase portrait of the circuit in the figure below for \( u(t) = 1, t > 0 \) and initial condition \( x(t = 0) = [0 \quad 0]^T \).

b. Sketch the phase portrait of the system with initial condition \( x_0 = [2 \quad 1]^T \), and input \( u(t) \) as found in part 2b above.
1. Given the discrete-time system

\[
    x(t + 1) = \begin{bmatrix}
        0 & 1 & 0 \\
        0 & 0 & 1 \\
        1 & 2 & 3 \\
    \end{bmatrix} x(t) + \begin{bmatrix}
        0 \\
        0 \\
        1 \\
    \end{bmatrix} u(t)
\]

design a state-feedback control law for \( u \) that guarantees \( x(t) \to 0 \) in finite time from any initial condition.

2. Consider the continuous-time LTI system

\[
    \dot{x}(t) = Ax(t) \quad x(t) \in \mathbb{R}^n
\]

and suppose there exists a matrix \( P = P^T > 0 \) and a constant \( \alpha \) such that

\[
    A^T P + PA \leq 2 \alpha P. \quad (1)
\]

(a) In which region of the complex plane can you conclude that the eigenvalues of \( A \) lie?

(b) Using (1) show that \( \|x(t)\| \leq K\|x(0)\|e^{\alpha t} \) for an appropriate \( K \).

3. Consider the second order LTI system

\[
    \dot{x}(t) = \begin{bmatrix}
        0 & 1 \\
        -1 & 0 \\
    \end{bmatrix} x(t) + \begin{bmatrix}
        0 \\
        1 \\
    \end{bmatrix} u(t).
\]

(a) Draw the phase portrait for \( x(t) \) when the input is zero.

(b) Construct a bounded input \( u(t) \) that drives \( x(t) \) unbounded.