1 Polynomial Functions of a Matrix (Cayley-Hamilton)

**Theorem 1** (Cayley-Hamilton). Consider a matrix $A \in \mathbb{C}^{n \times n}$ with characteristic polynomial $\hat{\chi}_A(s) \equiv \det(sI - A) = s^n + d_1 s^{n-1} + \cdots + d_n$, the characteristic polynomial of $A$. Then

$$\hat{\chi}_A(A) = A^n + d_1 A^{n-1} + \cdots + d_n I = 0 \quad \text{(characteristic equation)}$$

That is, any square matrix $A$ will satisfy its own characteristic equation.

**Proposition 2.** Every polynomial function $f$ of $A$ can be written as a function of $I, A, A^2, \cdots, A^{n-1}$.

**Proof.** We can divide the polynomial function $f$ in $s$ by the characteristic polynomial $\hat{\chi}_A(s)$ to obtain

$$f(s) = \hat{\chi}_A(s)q(s) + r(s),$$

where the order of the residue polynomial $r$ is no higher than $n-1$. Hence $f(A) = \hat{\chi}_A(A)q(A) + r(A) = r(A)$, which implies that $f(A)$ can be written as a function of $I, A, A^2, \cdots, A^{n-1}$. 

**Problem 1.** Recall the following problem from homework 1: Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Is the set $\{I, A, A^2\}$ linearly dependent or independent in $\mathbb{R}^{2 \times 2}$?

**Problem 2.** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

Using Proposition 2, compute $f_1(A) = A^3$. 

2 Diagonalization

Definition 3. Let $A \in \mathbb{C}^{n \times n}$. A vector $v_i \in \mathbb{C}^n$, $v_i \neq 0$ is called an eigenvector of $A$ if $Av_i = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$, the eigenvalue corresponding to $v_i$.

Definition 4. Suppose $v_1, \ldots, v_n$ is a linearly independent set of eigenvectors of $A \in \mathbb{R}^{n \times n}$. Then we can write $Av_i = \lambda_i v_i$, $i = 1, \ldots, n$. This can also be written as:

$$A [v_1 \cdots v_n] = [v_1 \cdots v_n] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix}$$

Define $T^{-1} = [v_1 \cdots v_n]$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then,

$$AT^{-1} = T^{-1} \Lambda$$

$$\Lambda = TAT^{-1}$$

Steps to diagonalize a matrix:

1. Compute the eigenvalues of the matrix; they are given by solutions of the characteristic polynomial.
2. Compute the (right) eigenvectors corresponding to each eigenvalue. Stacking these eigenvectors gives matrix $T^{-1}$. The diagonalization is now given by $T^{-1} \Lambda T$. 

Problem 3. In Problem 2, compute $f_2(A) = 2A^4 - 3A^3 - 3A^2 + 4I$.

Problem 4. In Problem 2, use Cayley-Hamilton to find $A^{-1}$. 

Consider. Not all matrices are diagonalizable.

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

Consider. If \( A \) is diagonalizable, does that mean it must have distinct eigenvalues?

Problem 5. Prove that Cayley-Hamilton holds for diagonalizable matrices

Problem 6. Let \( A \in \mathbb{R}^{n \times n} \) be diagonalizable. Using the dyadic expansion discussed in class (see Lecture notes 12), determine \( e^{At} \).
Problem 7. Consider $\dot{x} = Ax$ with $x_0 = v_i$, where $A \in \mathbb{R}^{2\times2}$ and $v_i$ is its eigenvector. Assume that all eigenvalues of $A$ are real. Find an expression for $x(t)$ in terms of $v_i, t$. Give a geometric interpretation of the trajectory of $x(t)$.

Problem 8. Let’s generalize the previous problem. Consider the system $\dot{x} = Ax$ with $x_0 \in \mathbb{R}^n$. Suppose that $A$ has $n$ linearly independent eigenvectors $v_1, \ldots, v_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Find an expression for $x(t)$ in terms of the eigenvalues and eigenvectors of $A$. 
Problem 9. This is a problem from Professor Ron Fearing on the Spring’14 prelim.

Given: \( A = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \)

1. Find the characteristic polynomial for \( A \)

2. Express \( A^4 \) in terms of the lowest order polynomial in \( A \)

3. Find \( e^{At} \) by Cayley-Hamilton; that is show that \( e^{At} = \alpha_0(t)I + \alpha_1(t)A \)