EE21A Problem Set 8 – Solutions

Problem 1. From the definition of the adjoint (LN4 pg7-8) we have \( \langle \mathcal{L}_o^*(y), x_0 \rangle_{\mathbb{R}^n} = \langle y, \mathcal{L}_o(x_0) \rangle_Y \). Manipulating the latter inner product we have
\[
\langle y, \mathcal{L}_o(x_0) \rangle_Y = \int_{t_0}^{t_1} y^*(\tau) C(\tau) \Phi(\tau, t_0) x_0 d\tau = \int_{t_0}^{t_1} y^*(\tau) C(\tau) \Phi(\tau, t_0) d\tau x_0
\]
\[
= \int_{t_0}^{t_1} (\Phi^*(\tau, t_0) C^*(\tau) y(\tau))^* d\tau x_0 = \left( \int_{t_0}^{t_1} \Phi^*(\tau, t_0) C^*(\tau) y(\tau) d\tau \right)^* x_0
\]
from which we see that \( \mathcal{L}_o^*(y) = \int_{t_0}^{t_1} \Phi^*(\tau, t_0) C^*(\tau) y(\tau) d\tau \).

Problem 2. Writing the controllability grammian we have: \( W_c[t_0', t_1'] = \int_{t_0'}^{t_1'} \Phi(t_1', \tau) B(\tau) B^*(\tau) \Phi^*(t_1', \tau) d\tau \). We can split the integral up into each component of time:
\[
W_c[t_0', t_1'] = \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) B^*(\tau) \Phi^*(t_0, \tau) d\tau + \int_{t_0}^{t_1} \ldots d\tau + \int_{t_1}^{t_1'} \ldots d\tau
\]
\[
= W_c[t_0', t_0] + W_c[t_0, t_1] + W_c[t_1, t_1']
\]
We know that each grammian is positive semidefinite (PSD) by default. We also know \( W_c[t_0, t_1] \) is c.c., so it is positive definite (PD) \( (x^T W_c[t_0, t_1] x > 0) \). To test \( W_c[t_0', t_1'] \) we are adding two PSD matrices with a PD matrix, making the result PD:
\[
x^T W_c[t_0', t_1'] x = x^T W_c[t_0, t_0] x + x^T W_c[t_0, t_1] x + x^T W_c[t_1, t_1'] x > 0
\]
Therefore, \( W_c[t_0', t_1'] \) is c.c.

To show the reverse is not true, show that a PD matrix does not necessarily split into a sum of PD matrices.

Problem 3. Note that the rank of \( sI - A \) drops whenever \( s \in \sigma(A) \). In particular, consider \( \lambda_1 \in \sigma(A) \). The “rank drop” from \( \lambda_1 I - A \) is given by the number of Jordan blocks associated with \( \lambda_1 \). Therefore, the maximum rank drop will be caused by the eigenvalue with the maximum number of Jordan blocks. In fact, the number of inputs necessary to ensure controllability is equal to the maximum number of Jordan blocks associated with an eigenvalue.

The maximum Jordan blocks associated with an eigenvalue \( \lambda_i \) is given by \( (d_i - m_i + 1) \), which corresponds to one jordan block of size \( m_i \) and rest each of size 1. Therefore, under the worst case, the minimum number of inputs is given by \( \max_i (d_i - m_i + 1) \).
In particular, consider the following example:

\[
A = \begin{bmatrix}
-3 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\] (1)

Suppose we want to find the minimum number of inputs needed for the system to be controllable. The maximum amount of “rank drop” from \( sI - A \) is two (corresponding to eigenvalue 1), so we need a \( B \) matrix with rank at least two (and thus at least two columns corresponding to two inputs). Since the number of inputs necessary to ensure controllability is equal to the maximum number of Jordan blocks associated with an eigenvalue. In this case at least two inputs are needed. It is easy to find a two-column \( B \) matrix that does the job, for example,

\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}.
\] (2)

Problem 4. Follow dual proof for controllability in LN 17 pg 11-14

Problem 5. With \( k = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \) the closed loop system is

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\alpha_3 & -\alpha_2 & -\alpha_1 \\
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix} \left( v - \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} x \right)
\]

\[
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\alpha_3 - k_1 & -\alpha_2 - k_2 & -\alpha_1 - k_3 \\
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix} v.
\]

Its characteristic polynomial is \( s^3 + (\alpha_3 + k_1)s^2 + (\alpha_2 + k_2)s + (\alpha_1 + k_3) \). For any desired characteristic polynomial we can set the \( k \)'s to match. Given our desired characteristic polynomial as \( p(s) = b_0s^3 + b_1s^2 + b_2s + b_3 \), we have \( (k_1, k_2, k_3) = \) For example, if we want \( s^3 + a_1s^2 + a_2s + a_3 \), we have \( (k_1, k_2, k_3) = (a_3 - \alpha_3, a_2 - \alpha_2, a_1 - \alpha_1) \).