Problem 1: Adjoint of Observability Map.

Given the observability map \( L_O: \mathbb{R}^n \rightarrow \mathcal{Y}_{[t_0, t_1]} \) where \( L_O(x_0) = C(\cdot)\Phi(\cdot, t_0)x_0 \), derive its adjoint map \( L_O^* \).

Problem 2: Controllability over time intervals.

Given a linear time varying system \( R(\cdot) = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)] \), show that if \( R(\cdot) \) is completely controllable on \([t_0, t_1]\), then \( R(\cdot) \) is completely controllable on any \([t'_0, t'_1]\), where \( t'_0 \leq t_0 < t_1 \leq t'_1 \). Show that this is no longer true when the interval \([t_0, t_1]\) is not a subset of \([t'_0, t'_1]\).

Problem 3: Controllability, characteristic and minimal polynomials.

Consider an LTI system \((A, B, C)\) with \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{n_o \times n} \). You are told that the characteristic polynomial of \( A \) is \((s - \lambda_1)^{d_1}(s - \lambda_2)^{d_2} \ldots (s - \lambda_k)^{d_k}\) and the minimal polynomial is \((s - \lambda_1)^{m_1}(s - \lambda_2)^{m_2} \ldots (s - \lambda_k)^{m_k}\). The ordering of the \( \lambda_i \) is chosen to be such that \( m_1 \leq m_2 \ldots \leq m_k \). Compute the minimum number of inputs required (that is the minimum size of \( n_i \)) so as to make the pair \((A, B)\) completely controllable. Similarly, compute the minimum value of \( n_o \) to make the pair \((A, C)\) completely observable.

Problem 4: Observability Tests for LTI Systems.

Consider the following theorem (page 14 of Lecture Notes 17):

**Theorem:** The LTI system represented by \((A, C)\) is completely observable on some \([0, \Delta]\) \(\iff\) rank

\[
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
= n
\]

\(\iff\) rank

\[
\begin{bmatrix}
sI - A \\
C
\end{bmatrix}
= n, \forall s \in \sigma(A)
\]

Prove the following 4 directions: \((a) \Rightarrow (b), (b) \Rightarrow (a), (b) \Rightarrow (c), \) and \((c) \Rightarrow (b)\).

One way to prove these is to consider the matrices \((A^T, C^T)\) and follow the controllability results directly (why does this work?).

Problem 5: A prequel question to controller design.

Consider the linear time invariant system with state equation:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\alpha_3 & -\alpha_2 & -\alpha_1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u
\]

Insert state feedback: the input to the overall closed loop system is \(v\) and \(u = v - kx\) where \(k\) is a constant.
row vector. Show that given any polynomial $p(s) = \sum_{k=0}^{3} b_k s^{3-k}$ with $b_0 = 1$, there is a row vector $k$ such that the closed loop system has $p(s)$ as its characteristic equation. (This naturally extends to $n$ dimensions, and implies that any system with a representation that can be put into the form above, called Controllable Canonical Form, can be stabilized by state feedback.)