1 Inner product spaces and Hilbert spaces

Definition 1. Let $x, y \in \mathbb{R}^n$ where $x = [x_1, \ldots, x_n]^T$ and $y = [y_1, \ldots, y_n]^T$. The dot product of $x$ and $y$, denoted $x \cdot y$ is defined by

$$x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

Definition 2. Consider a vector space $(H, \mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. The function $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{F}$ is called an inner product if

1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$, $\forall \alpha \in \mathbb{F}$
3. $\|x\|^2 = \langle x, x \rangle \geq 0$, and $\|x\|^2 = 0 \iff x = 0$  
   (here $\|x\|$ is referred to as the norm induced by the inner product)
4. $\langle x, y \rangle = \langle y, x \rangle$

Example. Let’s restrict ourselves to $\mathbb{R}^2$. One of the most common inner products on $\mathbb{R}^2$ is the dot product, but we can also consider the weighted inner product:

$$\langle x, y \rangle = 2x_1y_1 + 5x_2y_2$$

Problem 1. Verify that

$$\langle x, y \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$$

is an inner product on $\mathbb{R}^n$.

Definition 3. A space equipped with an inner product is called an inner product space.

Definition 4. A space equipped with a norm is called a normed space.

Definition 5. A space is complete if every Cauchy sequence of elements in that space converges to an element in that space. A Cauchy sequence is any sequence $x_1, x_2, \ldots$ such that

$$\forall \varepsilon > 0, \exists N \text{ such that } \|x_m - x_n\| < \varepsilon, \quad \forall m, n > N$$
**Definition 6.** A complete inner product space is called a *Hilbert space* (here completeness is w.r.t the norm induced by the inner product).

**Definition 7.** A complete normed space is called a *Banach space* (here completeness is w.r.t any norm).

**Proposition 8.** Some basic properties of an inner product:

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \overline{\alpha} \langle x, y \rangle$
3. $\langle \alpha x + \beta y, \gamma z + \delta w \rangle = \overline{\alpha} \gamma \langle x, z \rangle + \overline{\beta} \gamma \langle y, z \rangle + \overline{\alpha} \delta \langle x, w \rangle + \overline{\beta} \delta \langle y, w \rangle$

**Problem 2.** Prove the above properties.
2 Orthogonality

Let $H$ be a Hilbert space (remember: this means $H$ is a vector space with an inner product and the space is complete).

Definition 9. Given $x, y \in H$, $x$ and $y$ are said to be orthogonal if $\langle x, y \rangle = 0$.

Definition 10. $M^\perp := \{ y \in H | \langle x, y \rangle = 0, \forall x \in M \}$ is called the orthogonal complement of $M$.

Exercise. Prove that $M^\perp$ is a vector space.

Definition 11. Supposed $u, v \in H$ with $v \neq 0$. Let $c = \frac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \frac{\langle u, v \rangle}{\|v\|^2}v$. Then:

$$\langle w, u \rangle = 0 \text{ and } u = cv + w$$

Problem 3 (Cauchy-Schwarz inequality). Let $H$ be a Hilbert space. Show that

$$|\langle a, b \rangle| \leq \|a\|\|b\|$$
3 Adjoint

Definition 12. Let $U, V$ be Hilbert spaces, and $A : U \to V$ be a continuous linear map. The map $A^* : V \to U$ is said to be the adjoint of $A$ if

$$\langle v, A(u) \rangle_V = \langle A^*(v), u \rangle_U$$

Exercise. Prove that $A^*$ is a linear map.

Problem 4. Let $A : U \to V$ be a linear map, where $U, V$ are Hilbert spaces. Show that

$$\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$$