1 Singular Value Decomposition

Definition 1. A matrix \( M \in \mathbb{R}^{n \times n} \) is called orthogonal if all rows and columns of the matrix are mutually orthogonal. Moreover, if all rows and columns have a unit norm, the matrix is called orthonormal. So for an orthonormal matrix, we have \( M^T M = M M^T = I \), where \( I \) is an identity matrix of size \( n \times n \).

Theorem 2. Any \( m \times n \) matrix can be factored into \( A = U \Sigma V^\top \), where \( U \) is an \( m \times m \) orthogonal matrix, \( V \) is an \( n \times n \) orthogonal matrix, and \( \Sigma \) has the form

\[
\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}
\]

where \( \text{rk} A = r \) and \( \sigma_1, \ldots, \sigma_r \) are the singular values of \( A \).

To do the proof/construction of the SVD, we need the following result:

Lemma 3. The columns of \( U \) are orthonormal eigenvectors of \( AA^\top \), the columns of \( V \) are orthonormal eigenvectors of \( A^\top A \) and \( \sigma_i^2 \)'s are the eigenvalues of \( AA^\top \) (or \( A^\top A \)).

Consider. Why are the columns of \( U \) are orthonormal eigenvectors of \( AA^\top \)? How about the columns of \( V \) being orthonormal eigenvectors of \( A^\top A \)?
Problem 1. Find the SVD of

\[ A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \]
Consider. \textit{(Geometric Interpretation of SVD.)} Consider the matrix
\[
A = \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}
\]
with SVD
\[
U = \begin{bmatrix} -0.8507 & -0.5257 \\ -0.5257 & 0.8507 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 8.7134 & 0 \\ 0 & 3.3282 \end{bmatrix}, \quad V = \begin{bmatrix} -0.5946 & 0.8041 \\ -0.8041 & -0.5946 \end{bmatrix}
\]

How can we geometrically interpret the linear map $A$ through its SVD? Consider the unit circle and let’s see how the matrix can transform it.

(a) Unit circle and vectors.  
(b) Unit circle after applying $A$.

Figure 1: Visual representation of linear map $A$ acting on unit circle.

Let’s go step-by-step through how SVD decomposes this process into three transformations:

(a) Unit circle.  
(b) $V$ rotates.  
(c) Sigma scales.  
(d) $U$ rotates again.

Figure 2: Visualization of SVD.
Problem 2. Matrix 2-norm. Prove that

\[ \|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = \sigma_1 \]

Problem 3. Find the SVD of

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \]