Problem 1: Fields.
(a) Define addition and multiplication on \{0, 1\} to form a field. Show that your result is a field.
(b) Use the axioms of the field to show that, in any field, the additive identity and the multiplicative identity are unique.
(c) Is \(GL_n\), the set of all \(n \times n\) nonsingular matrices, a field? Justify your answer.

Problem 2: Functions. Consider \(f : \mathbb{R}^3 \to \mathbb{R}^3\), defined as
\[
f(x) = Ax, \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad x \in \mathbb{R}^3
\]

Problem 3: Functions, again. You work at a chemical plant where you need to keep track of the temperatures of three chemical reactors. Your coworker designed a user interface that takes in \(x \in \mathbb{R}^3\), where \(x_j\) is the temperature of reactor \(j\). The user interface then maps \(x\) through \(f(x)\), and outputs the values \(y \in \mathbb{R}^2\).
\[
y = f(x) = Ax
\]
\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\]
(a) Is \(f\) a function? What is the domain? What is the codomain? Is it injective? Is it surjective?
(b) Is this a well-designed interface? In other words, does it matter if this is a function/injective/surjective? Why or why not?

Problem 4: Vector Spaces.
(a) Show that \((\mathbb{R}^n, \mathbb{R})\), the set of all ordered \(n\)-tuples of elements from the field of real numbers \(\mathbb{R}\), is a vector space.
(b) Show that the set of all polynomials in \(s\) of degree \(k\) or less with real coefficients is a vector space over the field \(\mathbb{R}\). Find a basis. What is the dimension of the vector space?

Problem 5: Subspaces.
Is a plane in \(\mathbb{R}^3\) a subspace of \((\mathbb{R}^3, \mathbb{R})\)?

Problem 6: Subspaces.
Suppose \(U_1, U_2, \ldots, U_m\) are subspaces of a vector space \(V\). The sum of \(U_1, U_2, \ldots, U_m\), denoted \(U_1 + U_2 + \ldots + U_m\), is defined to be the set of all possible sums of elements of \(U_1, U_2, \ldots, U_m\):
\[
U_1 + U_2 + \ldots + U_m = \{u_1 + u_2 + \ldots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}
\]
(a) Is \(U_1 + U_2 + \ldots + U_m\) a subspace of \(V\)?
(b) Prove or give a counterexample: if $U_1, U_2, W$ are subspaces of $V$ such that $U_1 + W = U_2 + W$, then $U_1 = U_2$.

**Problem 7: Subspaces.** Consider the set of sequences $\{f_k\}_{k=0}^{\infty} := \{f_0, f_1, f_2, \ldots\}$ satisfying $f_k = f_{k-1} + f_{k-2}$ where $f_0$ and $f_1$ are arbitrary real numbers. Is this a subspace in the vector space of all sequences of real numbers over the field of real numbers?

**Problem 8: Subspaces.** Prove that the union of two subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces is contained in the other.

**Problem 9: Linear Independence.**

Let $V$ be the set of 2-tuples whose entries are complex-valued rational functions. Consider two vectors in $V$:

$$v_1 = \begin{bmatrix} 1/(s + 1) \\ 1/(s + 2) \end{bmatrix}, v_2 = \begin{bmatrix} (s + 2)/[((s + 1)(s + 3))] \\ 1/(s + 3) \end{bmatrix}$$

Is the set $\{v_1, v_2\}$ linearly independent over the field of rational functions? Is it linearly independent over the field of real numbers?

**Problem 10: Linear Independence.** Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Is the set $\{I, A, A^2\}$ linearly dependent or independent in $\mathbb{R}^{2\times2}$?

**Problem 11: Linear Independence.** Which of the following sets are linearly independent in $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

**Problem 12: Bases.** Let $U$ be the subspace of $\mathbb{R}^5$ defined by

$$U = \{[x_1, x_2, \ldots, x_5]^T \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}$$

Find a basis for $U$.

**Problem 13: Bases.** Prove that if $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in $V$, then so is the set $\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}$.

**Problem 14: Bases.** Consider $\mathbb{R}^n$ and let $e_i$ be the vector whose $i$th component is 1 and all other components are zero. Show that the span of $\{e_i\}_{i=1}^n$ is $\mathbb{R}^n$.

**Problem 15: Linearity.** Are the following maps $A$ linear?

(a) $A(u(t)) = u(-t)$ for $u(t)$ a scalar function of time

(b) How about $y(t) = A(u(t)) = \int_0^t e^{-\sigma} u(t - \sigma) d\sigma$?

(c) How about the map $A : as^2 + bs + c \to \int_0^t (bt + a) dt$ from the space of polynomials with real coefficients to itself?