Problem 1: Rank-Nullity Theorem. Let $A$ be a linear map from $U$ to $V$ with $\dim U = n$ and $\dim V = m$. Show that
\[ \dim R(A) + \dim N(A) = n \]

Problem 2: Properties of Linear maps. Consider the linear map $A : U \rightarrow V$. Suppose that $A(x_0) = b$. Then $A(x) = b$ if and only if $x - x_0 \in N(A)$.

Problem 3: Matrix Representation of a Linear Map, Change of Basis Let $A : (F^2, F) \rightarrow (F^3, F)$ be defined by $A(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$. What is the matrix representation of $A$ with respect to the standard bases, and with respect to new bases:

\[ B_{F^2} = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}, B_{F^3} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \]

Problem 4: Matrix Representation of a Linear Map. Let $A : (U, F) \rightarrow (V, F)$ with $\dim U = n$ and $\dim V = m$ be a linear map with $\text{rank}(A) = k$. Show that there exist bases $(u_i)_{i=1}^n$ and $(v_j)_{j=1}^m$ of $U, V$ respectively such that with respect to these bases $A$ is represented by the block diagonal matrix

\[ A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \]

What are the dimensions of the different blocks?

Problem 5: Matrix Representation of a Linear Map. From Callier and Desoer, p. 421

Let $A$ be a linear map of the $n$-dimensional linear space $(V, F)$ onto itself. Assume that for some $\lambda \in F$ and basis $(v_i)_{i=1}^n$ we have

\[ Av_1 = \lambda v_1 \]

and

\[ Av_k = \lambda v_k + v_{k+1} \quad k = 2, \ldots, n \]

Obtain a representation of $A$ with respect to this basis.

Problem 6: Sylvester’s Inequality. In class, we’ve discussed the Range of a linear map, denoting the rank of the map as the dimension of its range. Since all linear maps between finite dimensional vector spaces can be represented as matrix multiplication, the rank of such a linear map is the same as the rank of its matrix representation.

Given $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ show that

\[ \text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB) \leq \min \{ \text{rank}(A), \text{rank}(B) \} \]

Problem 7. Show that on $(F^n, F)$, the 1-norm, 2-norm, and $\infty$-norm are all equivalent.

Problem 8. Show that the induced matrix norm: $||A||_{\infty,i} = \max_{i=1\ldots m} \sum_{j=1}^n |a_{ij}|$. 