EE221A Linear System Theory
Problem Set 5
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Problem 1: Satellite Problem, linearization, state space model.
Model the earth and a satellite as particles. The normalized equations of motion, in an earth-fixed inertial frame, simplified to 2 dimensions (from Lagrange’s equations of motion, the Lagrangian \( L = T - V = \frac{1}{2}r^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{k}{r} \)):
\[
\ddot{r} = r\dot{\theta}^2 - \frac{k}{r} + u_1
\]
\[
\ddot{\theta} = -2\frac{\dot{r}}{r} + \frac{1}{r}u_2
\]
with \( u_1, u_2 \) representing the radial and tangential forces due to thrusters. The reference orbit with \( u_1 = u_2 = 0 \) is circular with \( r(t) \equiv p \) and \( \theta(t) = \omega t \). From the first equation it follows that \( p^3\omega^2 = k \). Obtain the linearized equation about this orbit.

Problem 2: State Transition Matrix, calculations.
Calculate the state transition matrix for \( \dot{x}(t) = A(t)x(t) \), with the following \( A(t) \):
\[
(a) \quad A(t) = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} ; \quad (b) \quad A(t) = \begin{bmatrix} -2t & 0 \\ 1 & -1 \end{bmatrix} ; \quad (c) \quad A(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix}
\]

Hint: for part (c) above, let \( \Omega(t) = \int_0^t \omega(t')dt' \); and consider the matrix
\[
\begin{bmatrix} \cos \Omega(t) & \sin \Omega(t) \\ -\sin \Omega(t) & \cos \Omega(t) \end{bmatrix}
\]

(d) For both of systems (a) and (b) above, describe the zero input (non-zero initial state) response.

Problem 3: Discrete-time time-invariant linear system (8 points).
You are given a linear, time-invariant system
\[
\dot{x} = Ax + Bu, \quad x(0) = x_0
\]
which is sampled every \( T \) seconds starting from \( t = 0 \). Denote \( x(kT) \) by \( x(k) \). Further, the input \( u \) is held constant between \( kT \) and \( (k+1)T \), that is, \( u(t) = u(k) \) for \( t \in [kT, (k+1)T) \).
(a) Suppose the system starts at time $kT$ with initial condition $x(kT)$. Using your knowledge of solutions to LTI systems, write down the solution to (1) at time $(k+1)T$, using the constant input $u(k)$ over the time interval $t \in [kT,(k+1)T]$. That is, solve for $A$ and $B$ in the state update equation for this discrete-time linear system

$$x(k + 1) = Ax(k) + Bu(k)$$

(b) Suppose that, in addition, the readout map of this discrete-time linear system is given by $y(k) = Cx(k) + Du(k)$. What is the response function of the discrete-time linear system, in terms of $k$, 0, $x(0)$ and the input sequence $\{u(0), \ldots, u(k)\}$?

**Problem 4: Discrete-time LQR.**

Consider the following optimal control problem where we are interested in controlling the output instead of state:

$$\min_U \sum_{\tau=0}^{N-1} (y_\tau^T Q y_\tau + u_\tau^T R u_\tau)$$

subject to

$$x_{t+1} = Ax_t + Bu_t, \quad t \in \{0,1,\ldots,N-1\} \quad (2)$$

$$y_t = Cx_t,$$

$$x_0 = x^{\text{init}}.$$ 

Here, $U$ is the sequence of control inputs as in lecture. Find an LQR-like sequence of matrix updates that computes the optimal cost-to-go at all times and the optimal feedback controller at all times.

**Problem 5: Discrete-time LQR.**

In the above problem, suppose the system matrices are given by:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and the cost matrices are given by $Q = Q_f = \rho_Q I$ and $R = \rho_R I$. Let $x_0 = (1,0)$ and $N = 20$. Explain how the output, control and cost-to-go change under the optimal feedback when (i) \(\rho_Q = 1, \rho_R = 1\), (ii) \(\rho_Q = 10^2, \rho_R = 1\) and (i i) \(\rho_Q = 1, \rho_R = 10^2\). You can use MATLAB, python, or whatever you like to solve the problem.

**Problem 6: Continuous-time LQR, infinite horizon.** Consider the system described by the equations $\dot{x} = Ax + Bu$, $y = Cx$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(a) Determine the optimal control $u^*(t) = F^*x(t), t \geq 0$ which minimizes the performance index $J = \int_0^\infty (y^2(t) + \rho u^2(t))dt$ where $\rho$ is positive and real.

(b) Observe how the eigenvalues of the dynamic matrix of the resulting closed loop system change as a function of $\rho$. Can you comment on the results?