Problem 1.

\[ A_1 = \begin{bmatrix} 6 & 5 & -1 \\ -1 & 0 & 1 \\ 3 & 3 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 5 & -2 \\ 3 & 4 & 0 \end{bmatrix} \]

**For matrix A1:** To find the eigenvalues of the matrix \( A_1 \), we need to find the roots of its characteristic equation by computing \( \det(sI - A_1) = 0 \).

\[ \hat{\chi}_{A_1}(s) = s^3 - 8s^2 + 17s - 10 = (s - 1)(s - 2)(s - 5) \]

We have 3 distinct eigenvalues \( \lambda = 1, 2, 5 \) so we know that this matrix is diagonalizable. To construct \( T^{-1} \), let’s find the eigenvectors for each eigenvalue:

For \( \lambda = 1 \), we solve \((A_1 - I)e_1 = 0\) to get \( e_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \).

For \( \lambda = 2 \), we solve \((A_1 - 2I)e_2 = 0\) to get \( e_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \).

For \( \lambda = 5 \), we solve \((A_1 - 5I)e_3 = 0\) to get \( e_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \).

Thus, we can construct the decomposition of matrix \( A_2 \) as

\[ A_2 = T^{-1}\Lambda T = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \]

**For matrix A2:** To find the eigenvalues of the matrix \( A_2 \), we need to find the roots of its characteristic equation by computing \( \det(sI - A_2) = 0 \).

\[ \hat{\chi}_{A_2}(s) = s^3 - 7s^2 + 15s - 9 = (s - 1)(s - 3)^2 \]

We do not have 3 distinct eigenvalues \( \lambda = 1, 2, 5 \) so we will use the Jordan decomposition. Let’s find the eigenvectors (and generalized eigenvectors) for each eigenvalue:

For \( \lambda = 1 \), we solve \((A_2 - I)e_1 = 0\) to get \( e_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \). Since \( e_1 \notin \mathcal{R}(A_2 - I) \), we cannot find a generalized eigenvector and we have the maximal Jordan chain.
For $\lambda = 3$, we solve $(A^2 - 3I)e_2 = 0$ to get $e_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. We now need to find a generalized eigenvector by solving $(A^2 - 3I)v_2 = e_2$ to get $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Thus, we can construct the Jordan decomposition of matrix $A^2$ as

\[ A^2 = T^{-1}JT = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \]

**Problem 2.** We know that the determinant of a matrix is equal to the product of its eigenvalues. We also know that $e^A$ is given by a convergent power series.

\[ e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \]

If we post-multiply this expression by $v$, an eigenvector of $A$ associated to eigenvalue $\lambda$, we obtain

\[ e^{Av} = \sum_{n=0}^{\infty} \frac{A^n v}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n v}{n!} = e^{\lambda} v \quad (1) \]

Because (1) holds for all eigenvectors of $A$, and the eigenvectors of a polynomial of a matrix are the eigenvalues of that matrix, the $i$-th eigenvalue of $e^A$ will be $e^{\lambda_i}$.

Any product of terms of the form $e^{\lambda_i}$ will not be equal to zero (they can only approach zero as $\sum_i c_i$ approaches $-\infty$). Therefore, the determinant of $e^A$ (which will be the product of eigenvalues $e^{\lambda_i}$) will not be equal to zero. This result is independent of the eigenvalues and the determinant of $A$.

**Problem 3.** You are told that $A : \mathbb{R}^n \to \mathbb{R}^n$ and that $R(A) \subset N(A)$. Are you able to determine $A$ up to a change of basis? No. The given property $R(A) \subset N(A)$ is equivalent to $A^2v = 0$, for all $v \in \mathbb{R}^n$. Clearly $A_0 = 0_{n \times n}$ satisfies this property, and so does

\[ A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots \\ 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \quad (2) \]

$A_0$ and $A_1$ are both in Jordan form, but are not the same (even with block reordering). They will have a different number of linearly independent eigenvectors (as well as different ranks) and will not be similar to one another (similarity transforms preserve matrix rank, as discussed in the solution for Problem 4). Hence, $A$ cannot be determined up to a change of basis (a change of basis is a similarity transformation).
Problem 4. We know that there is a single eigenvalue $\lambda = 0$ with algebraic multiplicity 6, and that the size of the largest Jordan block is 3. We know that $\text{rank}(A) = \text{rank}(T^{-1}JT) = \text{rank}(J)$ since $T$ is full rank (apply Sylvester’s inequality). Then $J$ must have rank of at least 2, arising from the ones on the superdiagonal in the Jordan block of size 3 (the diagonal entries will all be zeros since zero is the only eigenvalue). If all the other Jordan blocks were size 1, then there would be no additional ones on the superdiagonal, so the lower bound on rank $A$ is 2. Now the highest number of ones on the superdiagonal that this matrix could have is 4, which would be the case if there were two Jordan blocks of size 3. So rank $A \leq 4$. Thus the bounds are $2 \leq \text{rank}(A) \leq 4$.

Problem 5. A matrix $A$ has minimal polynomial $(s-\lambda_1)^2(s-\lambda_2)^3$. We want to find $\cos(e^A)$ as a polynomial in $A$. Interpolate a polynomial such that $\hat{p}(A) = f(A) = \cos(e^A)$, in which the spectral interpolation conditions (SIC) hold. The degree of the minimum polynomial is $m = 5$, and so it will be of degree 4. This will yield a $5 \times 5$ system that we need to solve. The interpolating polynomial will have the form:

$$
\hat{p}(s) = a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5
$$

We can write all the SIC conditions as:

$$
\begin{align*}
 a_1 \lambda_1^4 + a_2 \lambda_1^3 + a_3 \lambda_1^2 + a_4 \lambda_1 + a_5 &= \cos(e^{\lambda_1}) \\
 4a_1 \lambda_1^3 + 3a_2 \lambda_1^2 + 2a_3 \lambda_1 + a_4 &= -e^{\lambda_1} \sin(e^{\lambda_1}) \\
 a_1 \lambda_2^4 + a_2 \lambda_2^3 + a_3 \lambda_2^2 + a_4 \lambda_2 + a_5 &= \cos(e^{\lambda_2}) \\
 4a_1 \lambda_2^3 + 3a_2 \lambda_2^2 + 2a_3 \lambda_2 + a_4 &= -e^{\lambda_2} \sin(e^{\lambda_2}) \\
 12a_1 \lambda_2^2 + 6a_2 \lambda_2 + 2a_3 &= -e^{\lambda_2} [\sin(e^{\lambda_2}) + e^{\lambda_2} \cos(e^{\lambda_2})]
\end{align*}
$$

This is a system of linear equations that can be solved as:

$$
\Phi a = b \rightarrow a = \Phi^{-1} b
$$

in which:

$$
\Phi = \begin{bmatrix} 
\lambda_1^4 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\
4 \lambda_1^3 & 3 \lambda_1^2 & 2 \lambda_1 & 1 & 0 \\
\lambda_2^4 & \lambda_2^3 & \lambda_2^2 & \lambda_2 & 1 \\
4 \lambda_2^3 & 3 \lambda_2^2 & 2 \lambda_2 & 1 & 0 \\
12 \lambda_2^2 & 6 \lambda_2 & 2 & 0 & 0
\end{bmatrix}, \quad
a = \begin{bmatrix} 
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}, \quad
b = \begin{bmatrix} 
\cos(e^{\lambda_1}) \\
-e^{\lambda_1} \sin(e^{\lambda_1}) \\
\cos(e^{\lambda_2}) \\
-e^{\lambda_2} \sin(e^{\lambda_2}) \\
-e^{\lambda_2} [\sin(e^{\lambda_2}) + e^{\lambda_2} \cos(e^{\lambda_2})]
\end{bmatrix}
$$

Then, by solving $a$, $\cos(e^A)$ can be written as:

$$
\cos(e^A) = \hat{p}(A) = a_1 A^4 + a_2 A^3 + a_3 A^2 + a_4 A + a_5 I
$$

Please note that this approach require to solve a system of equations to determine the coefficients $a_i$. The matrix $\Phi$ do not depend on $f$, since that information is contained in the vector $b$.

Problem 6. We know that the characteristic polynomial of $A$ is given by $\tilde{\chi}_A(s) = (s-\lambda_1)^5(s-\lambda_2)^3$, and its minimum polynomial is given by $\psi_A(s) = (s-\lambda_1)^2(s-\lambda_2)^3$.

From this, we know that the largest block of the Jordan matrix for $\lambda_1$ is $2 \times 2$, and for $\lambda_2$ is $3 \times 3$. We also know that $\lambda_1$ has an algebraic multiplicity of 5 and $\lambda_2$ of 3. Then:
• The eigenvalue $\lambda_2$ will have only 1 block of $3 \times 3$, that contains only 1 eigenvector of $A$ and two generalized eigenvectors.

• The eigenvalue $\lambda_1$ will have a block of $2 \times 2$, that contains only 1 eigenvector of $A$ and one generalized eigenvector.

• We know two eigenvectors for the eigenvalue $\lambda_1$ that we need to fill in the last $3 \times 3$ block of the Jordan matrix. Since the maximum size for the blocks for this eigenvalue is $2 \times 2$, we only have one possibility to fill the Jordan Matrix. This is one block of $2 \times 2$ with one eigenvector and one generalized eigenvector, and the final block of $1 \times 1$ with only one eigenvector.

With that the matrix will have the form of:

$$J = \begin{bmatrix}
\lambda_2 & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 1 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1 & 1 & 0 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1 \\
\end{bmatrix}$$

and then

$$\cos(e^J) = \begin{bmatrix}
\cos(e^{\lambda_2}) & -e^{\lambda_2} \sin(e^{\lambda_2}) & -e^{\lambda_2} \left[\sin(e^{\lambda_2}) + e^{\lambda_2} \cos(e^{\lambda_2})\right]/2 & 0 & 0 & 0 & 0 \\
0 & \cos(e^{\lambda_2}) & -e^{\lambda_2} \sin(e^{\lambda_2}) & 0 & 0 & 0 & 0 \\
0 & 0 & \cos(e^{\lambda_2}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(e^{\lambda_1}) & -e^{\lambda_1} \sin(e^{\lambda_1}) & 0 & 0 \\
0 & 0 & 0 & 0 & \cos(e^{\lambda_1}) & -e^{\lambda_1} \sin(e^{\lambda_1}) & 0 \\
0 & 0 & 0 & 0 & 0 & \cos(e^{\lambda_1}) & \end{bmatrix}$$

and so:

$$\cos(e^A) = T \cos(e^J)T^{-1}$$

in which:

$$T = [e_1 \ v_1 \ w_1 \ e_2 \ v_2 \ e_3 \ v_3 \ e_4]$$

where $e_1$ is the eigenvector associated to $\lambda_2$ and $v_1$ and $w_1$ are the generalized eigenvectors associated to $\lambda_2$. $e_2$, $e_3$ and $e_4$ are the eigenvectors associated to $\lambda_1$, and $v_2$ and $v_3$ are the generalized eigenvectors associated to $\lambda_1$.

**Problem 7.** A being nonsingular means that it has all nonzero eigenvalues. This means that its jordan form has nonzero elements on the diagonal. Our function $f(x) = \cos(\log(x))$ is analytic everywhere except for at 0. Since there are no 0 eigenvalues in $A$, $\cos(\log(\lambda_i))$ and its derivatives will be well defined for all $i$. We can therefore write our analytic function as a polynomial function of $A$: $\cos(\log(A)) = p(A)$. 

November 24, 2019 4 / 5
Since \( x \in \mathcal{N}(\cos(\log(A))) \) we know \( \cos(\log(A))x = p(A)x = 0 \).

If this is an \( A \)-invariant subspace then \( Ax \in \mathcal{N}(\cos(\log(A))) \)

This would mean that \( \cos(\log(A))Ax = p(A)Ax = 0 \)

Polynomial functions of \( A \) can commute with \( A \), so this is equivalent to saying:

\[
\cos(\log(A))Ax = Ap(A)x = A0 = 0
\]

Therefore \( Ax \in \mathcal{N}(\cos(\log(A))) \), and so is an \( A \)-invariant subspace.

**Problem 8.** Let \( v \in V := \text{span} \{ b, Ab, A^2b, \ldots, A^{n-1}b \} \). Then \( v = \alpha_0b + \alpha_1Ab + \cdots + \alpha_{n-1}A^{n-1}b \) for some set of coefficients \( \{\alpha_i\}_{i=0}^{n-1} \in \mathbb{F}^n \). Thus

\[
Av = \alpha_0Ab + \alpha_1A^2b + \cdots + \alpha_{n-2}A^{n-1}b + \alpha_{n-1}A^n b
\]

From the Cayley-Hamilton Theorem, we have \( A^n = \beta_0I + \beta_1A + \cdots + \beta_{n-1}A^{n-1} \). Therefore

\[
Av = (\alpha_{n-1}\beta_0)b + (\alpha_0 + \alpha_{n-1}\beta_1)Ab + (\alpha_1 + \alpha_{n-1}\beta_2)A^2b + \cdots + (\alpha_{n-2} + \alpha_{n-1}\beta_{n-1})A^{n-1}b
\]

\[
= \gamma_0b + \gamma_1Ab + \cdots + \gamma_{n-1}A^{n-1}b
\]

and so \( Av \in V \). Hence, \( V \) is an \( A \)-invariant subspace.