Problem 1. (a) Choosing $x_1 = T_C$ and $x_2 = T_H$ we have $\dot{x} = Ax + Bu$ and $y = x$, where

$$ A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} $$

after plugging in numerical values for the constants. We find the eigenvalues as $\lambda_1 = -\frac{1}{10}$ and $\lambda_2 = -\frac{1}{2}$, with associated eigenvectors $v_1 = [1 \ 1]^\top$ and $v_2 = [1 \ -1]^\top$. We have $A = T^{-1}JT$, where $T^{-1} = [v_1 \ v_2]$ and $J = \text{diag}(\lambda_1, \lambda_2)$.

Define $z = Tx$. Then

$$ \dot{z} = Jz + TBu $$
$$ y = T^{-1}z $$

where $TB = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(b) We find that

$$ y(t) = T^{-1}z(t) = T^{-1}e^{Jt}z_0 = T^{-1}e^{Jt}Tx_0 = \frac{1}{2} \begin{bmatrix} e^{-0.1t} + e^{-0.5t} & e^{-0.1t} - e^{-0.5t} \\ e^{-0.1t} - e^{-0.5t} & e^{-0.1t} + e^{-0.5t} \end{bmatrix} x_0 $$

(c) The system has both eigenvalues in the open left half plane, and thus is internally exponentially stable. Therefore it is also BIBO stable (see Lecture Notes 15, page 7).

Problem 2. The transfer function has poles at $\pm j$. Hence, the poles are not in the open left half plane. So the system is not BIBO stable (see Lecture Notes 14, page 7). Consider input $u(t) = \cos(t)$, which is a bounded input that leads to $y(t) = \frac{1}{2}t\sin(t)$, which is unbounded over time.

Problem 3. (i) Both systems are linear time-invariant systems. We note that $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, is already in Jordan form, with non-distinct eigenvalues $\lambda = 0, 0$, each of which have a Jordan-block of size 1. Thus, even though the eigenvalues are on the $j\omega$-axis, since they each have a Jordan-block of size 1, then we know this system is internally stable at the origin.

(ii) For the second matrix, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, it has non-distinct eigenvalues $\lambda = 0, 0$ with Jordan-blocks of size $2 \times 2$. Thus, this system is not stable at the origin.

Problem 4. (a) Defining $z := [\delta x \ \delta v \ \delta \psi]^\top$ we have $\dot{z} = Az + Bu$, where

$$ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} E_A \\ E_G \end{bmatrix} $$
From this we compute $\hat{\chi}_A(s) = s(s^2 + g/R)$, which shows that the system is internally stable. W.l.o.g. we may consider a single output, so suppose $C = [c_1 \ c_2 \ c_3]$. The transfer function matrix is $H(s) = C(sI - A)^{-1}B$ is

$$H(s) = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \frac{1}{s(s^2 + \frac{g}{R})} \begin{bmatrix} * & * & * \ s & s^2 & \frac{s^2}{R} & 1 & 0 \\ -g & -sg & s^2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{c_1 s^2}{s^2 + \frac{g}{R}} \ c_2 & c_3 \end{bmatrix}$$

We know that the system is BIBO stable if and only if $H(s)$ has no poles outside the open left half plane. But it is clear from the first element $H_1(s) = \frac{c_1 s^2}{s^2 + \frac{g}{R}}$ that the poles at $s = \pm j\sqrt{g/R}$ cannot be cancelled, so the only way for them to not show up in $H_1(s)$ is if $c_1 = c_2 = c_3 = 0$. Thus the system is BIBO stable only for the trivial output $y = 0$. We also know this because this is a time-invariant system and all eigenvalues of $A$ are on the $j\omega$-axis, no matter what $C$ we choose our transfer function will have poles on the $j\omega$-axis.

(b) With $C = [1 \ 0 \ 0]$ and $B = [0 \ 0 \ 1]^\top$, assuming zero initial condition, and $U(s) = \mathcal{L}(u(t)) = \frac{E_G}{s}$ we find

$$\delta X(s) = H(s)U(s) = C(sI - A)^{-1}B \frac{E_G}{s} = -\frac{-g}{s(s^2 + \frac{g}{R})} \frac{E_G}{s} = -gE_G \frac{1}{s^2(s^2 + \frac{g}{R})}$$

Partial fraction expansion yields

$$\frac{g}{s^2(s^2 + \frac{g}{R})} = \frac{R}{s^2} - \frac{R}{s^2 + \frac{g}{R}}$$

and so, transforming back to the time domain, we have

$$\delta x(t) = E_G R (-t + \sqrt{R/g} \sin \sqrt{g/R} t)$$

Hence the position error keeps growing (in absolute value), and oscillates around $\delta \bar{x} = -E_G R t$ with frequency $\sqrt{g/R}$.

Problem 5. We can rewrite this equation as

$$A^T P - \lambda P + PA - \lambda P = -Q$$

$$(A^T - \lambda I)P + P(A - \lambda I) = -Q$$

$$(A - \lambda I)^T P + P(A - \lambda I) = -Q$$

Now, based on the information given in the problem, we can apply the Lyapunov Theorem to the matrix $(A - \lambda I)$, indicating that its eigenvalues lie in the open left half-plane. We can see that the eigenvalues of
$(A - \lambda I)$ are equal to the eigenvalues of $A$ minus $\lambda$ (all vectors are eigenvectors of $\lambda I$, and if $Av = \alpha v$, then $(A - \lambda I)v = (a - \lambda)v$). Since the real part of all eigenvalues $\mu$ of $(A - \lambda I)$ are less than zero ($\text{Re}(\mu) < 0$), we can conclude that the real part of all eigenvalues $a$ of $A$ are less than $\lambda$ ($\text{Re}(a) < \lambda$).

Note: This form of the Lyapunov equation/theorem applies to real-valued $A$ matrices, which is generally the case for dynamic systems.