Nonlinear Systems

\[
\dot{x} = Ax + Bu \quad \rightarrow \quad \dot{x} = f(x, u) \quad (1)
\]

Analysis:

\[
\begin{align*}
\dot{x} &= f(x) \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{time-invariant (autonomous)} \\
\dot{x} &= f(t, x) \quad f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{time-varying (non-autonomous)}
\end{align*}
\]

Design:

\[
\dot{x} = f(x, u) \quad u \text{ to be designed as a function of } x.
\]

Equilibria

\[x = x^* \text{ is an equilibrium for } \dot{x} = f(x) \text{ if } f(x^*) = 0.\]

Example: Linear system \(\dot{x} = Ax\).

If \(A\) is nonsingular, \(x^* = 0\) is the unique equilibrium.

If \(A\) is singular, the nullspace defines a continuum of equilibria.

Example: Logistic growth model in population dynamics

\[
\dot{x} = f(x) = r \left(1 - \frac{x}{K}\right)x, \quad r > 0, \quad K > 0 \quad (2)
\]

\(x > 0\) denotes the population and \(K\) is called the carrying capacity.

For systems with a scalar state variable \(x \in \mathbb{R}\), stability can be determined from the sign of \(f(x)\) around the equilibrium. In this example \(f(x) > 0\) for \(x \in (0, K)\), and \(f(x) < 0\) for \(x > K\); therefore

\[
\begin{align*}
x &= 0 \quad \text{unstable equilibrium} \\
x &= K \quad \text{asymptotically stable.}
\end{align*}
\]
Linearization

Local stability properties of $x^*$ can be determined by linearizing the vector field $f(x)$ at $x^*$:

$$f(x^* + \tilde{x}) = f(x^*) + \left. \frac{\partial f}{\partial x} \right|_{x=x^*} \tilde{x} + \text{higher order terms} \quad (3)$$

Thus, the linearized model is:

$$\dot{\tilde{x}} = A\tilde{x}. \quad (4)$$

If $\Re\lambda_i(A) < 0$ for each eigenvalue $\lambda_i$ of $A$, then $x^*$ is asymptotically stable.

If $\Re\lambda_i(A) > 0$ for some eigenvalue $\lambda_i$ of $A$, then $x^*$ is unstable.

Example: Logistic growth model above:

![Logistic growth model graph]

Caveats:

1. Only local properties can be determined from the linearization.
   Example: The logistic growth model linearized at $x = 0$ ($\dot{x} = rx$) would incorrectly predict unbounded growth of $x(t)$. In reality, $x(t) \to K$.

2. If $\Re\lambda_i(A) \leq 0$ with equality for some $i$, then linearization is inconclusive as a stability test. Higher order terms determine stability.

   **Example:**
   
   $f(x) = x^3$ vs. $f(x) = -x^3$

   ![Comparison graphs]

   $f'(0) = 0$ in each case, but one is stable and the other is unstable.
Second order example: Pendulum

\[ \ell m \ddot{\theta} = -k \ell \dot{\theta} - mg \sin \theta \]  

(5)

Define \( x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \). State space: \( S^1 \times \mathbb{R} \).

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m} x_2 - \frac{g}{\ell} \sin x_1
\end{align*}
\]  

(6)

Equilibria: \((0, 0)\) and \((\pi, 0)\)

\[
\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} \cos x_1 & -\frac{k}{\ell} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{\ell} \end{bmatrix} \quad \text{(stable) at } x_1 = 0
\]

\[
\begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{\ell} \end{bmatrix} \quad \text{(unstable) at } x_1 = \pi
\]

Phase portrait: plot of \( x_1(t) \) vs. \( x_2(t) \) for 2nd order systems

Figure 1: Phase portrait of the pendulum for the undamped case \( k = 0 \).

Essentially Nonlinear Phenomena

1. Finite Escape Time

Example: \( \dot{x} = x^2 \)

\[
\frac{d}{dt} x^{-1} = -x^{-2} \dot{x} = -1
\]

\[
\Rightarrow \quad \frac{1}{x(t)} - \frac{1}{x(0)} = -t
\]

\[
\Rightarrow \quad x(t) = \frac{1}{\frac{1}{x(0)} - t}
\]

(7)

For linear systems, \( x(t) \to \infty \) cannot happen in finite time.
2. Multiple Isolated Equilibria

Linear systems: either unique equilibrium or a continuum
Pendulum: two isolated equilibria (one stable, one unstable)
“Multi-stable” systems: two or more stable equilibria

Example: bistable switch

\[
\begin{align*}
\dot{x}_1 &= -ax_1 + x_2 \\
\dot{x}_2 &= \frac{x_2^2}{1+x_1^2} - bx_2
\end{align*}
\]

\(x_1\): concentration of protein
\(x_2\): concentration of mRNA

(8)

\(a > 0, b > 0\) are constants. State space: \(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\).

This model describes a positive feedback where the protein encoded by a gene stimulates more transcription via the term \(\frac{x_2^2}{1+x_1^2}\).

Single equilibrium at the origin when \(ab > 0.5\). If \(ab < 0.5\), the line where \(\dot{x}_1 = 0\) intersects the sigmoidal curve where \(\dot{x}_2 = 0\) at two other points, giving rise to a total of three equilibria:

![Diagram showing three equilibria: one stable (gene off), one unstable (saddle point), and one stable (gene on).]