Review of Sum of Squares (SOS) Polynomials

Checking whether a polynomial is SOS: A polynomial \( p \) with degree \( \leq 2d \) is a sum of squares if and only if there exists \( Q = Q^T \geq 0 \) s.t.

\[
p(x) = z(x)^T Q z(x)
\]  

(1)

where \( z(x) \) is the vector of all monomials of degree \( \leq d \):

\[
z(x) \triangleq [1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^d]^T.
\]

Find a particular solution \( Q_0 \) such that

\[
p(x) = z(x)^T Q_0 z(x),
\]

and find a basis of symmetric matrices \( N_j, j = 1, 2, \ldots, K \), such that

\[
z(x)^T N_j z(x) = 0 \quad \text{for all } x.
\]

(2)

Then \( p \) is SOS if and only if there exist reals \( \lambda_1, \ldots, \lambda_K \) such that

\[
Q = Q_0 + \sum_{j=1}^{K} \lambda_j N_j \geq 0.
\]

(3)

This is a linear matrix inequality (LMI) and can be solved numerically with standard semidefinite program (SDP) solvers.

Synthesizing SOS Polynomials: Given \( p_i, i = 0, 1, \ldots, m \), each with degree \( \leq 2d \), find reals \( a_1, \ldots, a_m \) s.t. \( p_0 + a_1 p_1 + \cdots + a_m p_m \) is SOS.

Find a particular \( Q_i \) satisfying \( p_i = z^T Q_i z \) for each \( i = 0, 1, \ldots, m \).

Then search for \( a_1, \ldots, a_m \) and \( \lambda_1, \ldots, \lambda_K \) satisfying the LMI

\[
Q_0 + \sum_{i=1}^{m} a_i Q_i + \sum_{j=1}^{K} \lambda_j N_j \geq 0.
\]

(4)

Applications

Searching for a Lyapunov Function

Given \( \dot{x} = f(x), f(0) = 0 \), where \( f \) is a vector of polynomials, search for a Lyapunov function of the form

\[
V(x) = p_0(x) + a_1 p_1(x) + \cdots + a_m p_m(x)
\]

(5)
where \( p_i, i = 0, 1, \cdots, m \) are basis polynomials selected ahead of time, and \( a_i, i = 1, \cdots, m \) are weights to be determined.

To ensure \( V \) is positive definite, pick a positive definite polynomial \( \ell \) (e.g., \( \ell(x) = \epsilon x^T x \) for some small \( \epsilon \)) and impose the constraint:

\[
V(x) - \ell(x) \text{ is SOS.} \quad (6)
\]

To ensure \( \nabla V(x)^T f(x) \) is negative semidef., impose the constraint:

\[
-\nabla V(x)^T f(x) \text{ is SOS.} \quad (7)
\]

Constraints (6) and (7) can be brought to the LMI form (4) and feasible \( a_i, i = 1, \cdots, m \) can be determined numerically (if they exist).

Overapproximating Reachable Sets

Recall from Lecture 16 that

\[
R_T = \left\{ x(T) \mid \dot{x} = f(x, u), \ x(0) = 0, \int_0^T u^T(t)u(t)dt \leq 1 \right\} \quad (8)
\]

defines the reachable set from \( x(0) = 0 \) under unit energy inputs and, if we can find a positive definite \( V \) such that

\[
\nabla V(x)^T f(x, u) \leq u^T u, \quad (9)
\]

then we can overapproximate \( R_T \) by:

\[
R_T \subset \{ x : V(x) \leq 1 \}.
\]

This follows because, from (9),

\[
\frac{d}{dt} V(x(t)) \leq u^T u \Rightarrow V(x(T)) - V(x(0)) \leq \int_0^T u^T(t)u(t)dt \leq 1 \Rightarrow V(x(T)) \leq 1.
\]

If \( f(x, u) \) is a vector of polynomials in \( x \) and \( u \), we can search for a polynomial \( V \) of the form (5), and encode (9) with the constraint:

\[
-\nabla V(x)^T f(x, u) + u^T u \text{ is SOS in } x \text{ and } u. \quad (10)
\]

This can then be combined with (6) and brought to the LMI form (4).

Certifying Safety

If unsafe set \( U \) does not intersect the overapproximation above, then it can’t intersect the actual reachable set. Thus, we can certify safety by proving the implication:

\[
x \in U \Rightarrow V(x) \geq 1 + \epsilon \quad (11)
\]
for some $\varepsilon > 0$.

Suppose the unsafe set can be expressed as

$$U = \{x : q_i(x) \geq 0, \ i = 1, \cdots, p\}$$

where $q_i$ are polynomials. Then we can encode (11) with the constraints:

$$V(x) - (1 + \varepsilon) - \sum_{i=1}^{p} s_i(x)q_i(x) \text{ is SOS} \quad (12)$$

$$s_i(x), \ i = 1, \cdots, p \text{ are SOS.} \quad (13)$$

We can parameterize the search space for $s_i$ as we did for $V$ in (5), and combine (6), (10), (12)-(13) into a LMI.

Above we implicitly used a generalization of the S-procedure from Lecture 16. Specifically, to prove that $q_0(x) \geq 0$ whenever $q_i(x) \geq 0, \ i = 1, 2, \ldots, p$ we look for nonnegative functions $s_1, s_2, \ldots, s_p$ (rather than constants as in Lecture 16) such that

$$q_0(x) - \sum_{i=1}^{p} s_i(x)q_i(x) \geq 0.$$

**Underapproximating the Region of Attraction**

Given system $\dot{x} = f(x)$ with asymptotically stable equilibrium at the origin $x = 0$, the region of attraction, denoted $R_A$, is the set of initial conditions from which the trajectories converge to the origin.

Recall from Lecture 10 that, if $V$ is positive definite and

$$\nabla V(x)^T f(x) < 0 \quad \text{whenever } x \neq 0 \text{ and } V(x) \leq \gamma, \quad (14)$$

then $\Omega_\gamma = \{x : V(x) \leq \gamma\} \subset R_A$.

Let $\ell$ be a positive definite polynomial. If there exists a SOS polynomial $s$ such that

$$-[\ell(x) + \nabla V(x)^T f(x)] - s(x)[\gamma - V(x)] \text{ is SOS,} \quad (15)$$

then $V(x) \leq \gamma$ implies $\nabla V(x)^T f(x) \leq -\ell(x)$ as stipulated in (14).

To obtain a LMI from (15), one option is to fix the Lyapunov function $V$ and to parameterize the search space for $s$. We can further maximize $\gamma$ subject to (15) by incrementing $\gamma$ until the resulting LMI is infeasible.

Alternatively $s$ can be fixed and $V$ parameterized. If we parameterize both $s$ and $V$, however, (15) is no longer affine in the parameters because the term $s(x)V(x)$ contains the products of these parameters.
Below is a procedure that alternates between first fixing $V$, varying $s$, and next fixing $s$, varying $V$. When a new $V$ is obtained, however, the shape of the level set changes and it may be ambiguous whether the new one is bigger. To remove this ambiguity we define a "shape function" $p$ and use its level sets to judge the size of the region of attraction estimate.

Step 1: Let $V_0(x)$ be an initial choice for a Lyapunov function, e.g., a quadratic function for the linearized model at the origin. Find $\gamma^* := \max \gamma \text{ s.t. } \nabla V_0(x)^T f(x) < 0 \text{ whenever } x \neq 0 \text{ and } V_0(x) \leq \gamma$.

To satisfy the constraint look for a SOS multiplier $s_1(x)$ that satisfies

$$-[\ell(x) + \nabla V_0(x)^T f(x)] - s_1(x)[\gamma - V_0(x)]$$

is SOS

where $\ell$ is positive definite, e.g., $\ell(x) := \epsilon (x_1^2 + x_2^2)$ for some $\epsilon > 0$.

Step 2: Let $p(x)$ be some fixed, positive definite convex polynomial (e.g., $p(x) = x_1^2 + x_2^2$), and let $V_0(x)$ and $\gamma^*$ be as in Step 1. Find $\beta^* := \max \beta \text{ s.t. } V_0(x) \leq \gamma^* \text{ whenever } p(x) \leq \beta$.

To satisfy the constraint look for a SOS multiplier $s_2(x)$ such that

$$[\gamma^* - V_0(x)] - s_2(x)[\beta - p(x)]$$

is SOS.

This means that $\{x : p(x) \leq \beta\}$ is contained in $\{x : V_0(x) \leq \gamma^*\}$.

Step 3: Given $\gamma^*, s_1(x)$ from Step 1 and $p(x), s_2(x)$ from Step 2, search for $V(x)$ to solve:

$$\max_{\beta > 0, 4\text{th-order } V(x)} \beta$$

subject to

- $V(x) - \ell(x)$ is SOS
- $-[\ell(x) + \nabla V(x)^T f(x)] - s_1(x)[\gamma^* - V(x)]$ is SOS
- $[\gamma^* - V(x)] - s_2(x)[\beta - p(x)]$ is SOS.

The first constraint ensures $V$ is positive definite. The second implies that the level set $\{x : V(x) \leq \gamma^*\}$ is invariant, hence a valid approximation for the region of attraction. The third constraint and the maximization of $\beta$ ensure that $V$ is selected such that the level set $\{x : V(x) \leq \gamma^*\}$ is as large as possible, as measured by function $p$.

To proceed, replace $V_0(x)$ in Step 1 with the function $V(x)$ from Step 3, and repeat the steps above for several iterations, until the change in $\beta^*$ in Step 2 is sufficiently small. The final approximation of the ROA is the set where $V(x) \leq \gamma^*$. 