Essentially Nonlinear Phenomena Continued

1. Finite escape time
2. Multiple isolated equilibria
3. Limit cycles: Linear oscillators exhibit a continuum of periodic orbits; e.g., every circle is a periodic orbit for \( \dot{x} = Ax \) where

\[
A = \begin{bmatrix}
0 & -\beta \\
\beta & 0
\end{bmatrix} \quad (\lambda_{1,2} = \mp j\beta).
\]

In contrast, a limit cycle is an isolated periodic orbit and can occur only in nonlinear systems.

Example: van der Pol oscillator

\[
\begin{align*}
C\ddot{v}_C &= -i_L + v_C - \dot{v}_C^3 \\
L\dot{i}_L &= v_C
\end{align*}
\]

Example: Lorenz system (derived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere):

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz.
\end{align*}
\]

Chaotic behavior with \(\sigma = 10, b = 8/3, r = 28:\)

- For continuous-time, time-invariant systems, \(n \geq 3\) state variables required for chaos.
  - \(n = 1\): \(x(t)\) monotone in \(t\), no oscillations:

- \(n = 2\): Poincaré-Bendixson Theorem (to be studied in Lecture 3) guarantees regular behavior.
- Poincaré-Bendixson does not apply to time-varying systems and \(n \geq 2\) is enough for chaos (Homework problem).
- For discrete-time systems, \(n = 1\) is enough (we will see an example in Lecture 5).

Planar (Second Order) Dynamical Systems

Phase Portraits of Linear Systems: \(\dot{x} = Ax\)

- Distinct real eigenvalues

\[
T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\]
In $z = T^{-1}x$ coordinates:

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2.$$  

The equilibrium is called a node when $\lambda_1$ and $\lambda_2$ have the same sign (stable node when negative and unstable when positive). It is called a saddle point when $\lambda_1$ and $\lambda_2$ have opposite signs.

- Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$

$$T^{-1}AT = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\dot{z}_1 = \alpha z_1 - \beta z_2$$

$$\dot{z}_2 = \alpha z_2 + \beta z_1$$

$\rightarrow$ polar coordinates $\rightarrow \dot{r} = \alpha r$  

$$\dot{\theta} = \beta$$

Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

**hyperbolic equilibrium**: linearization has no eigenvalues on the imaginary axis

Phase portraits of nonlinear systems near hyperbolic equilibria are qualitatively similar to the phase portraits of their linearization. According to the Hartman-Grobman Theorem (below) a “continuous deformation” maps one phase portrait to the other.
Hartman-Grobman Theorem: If $x^*$ is a hyperbolic equilibrium of $\dot{x} = f(x), x \in \mathbb{R}^n$, then there exists a homeomorphism $z = h(x)$ defined in a neighborhood of $x^*$ that maps trajectories of $\dot{x} = f(x)$ to those of $\dot{z} = Az$ where $A \triangleq \left. \frac{\partial f}{\partial x} \right|_{x=x^*}$.

The hyperbolicity condition can’t be removed:

Example:

$$
\begin{align*}
\dot{x}_1 &= -x_2 + ax_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= x_1 + ax_2(x_1^2 + x_2^2)
\end{align*}
\Rightarrow \quad \dot{r} = ar^3
$$

$$
\dot{\theta} = 1
$$

$x^* = (0, 0)$  \quad $A = \left. \frac{\partial f}{\partial x} \right|_{x=x^*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

There is no continuous deformation that maps the phase portrait of the linearization to that of the original nonlinear model:

Periodic Orbits in the Plane

Bendixson’s Theorem: For a time-invariant planar system

$$
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
$$

if $\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region $D$, then there are no periodic orbits lying entirely in $D$. 
Proof: By contradiction. Suppose a periodic orbit $J$ lies in $D$. Let $S$ denote the region enclosed by $J$ and $n(x)$ the normal vector to $J$ at $x$. Then $f(x) \cdot n(x) = 0$ for all $x \in J$. By the Divergence Theorem:

$$\int_J f(x) \cdot n(x) d\ell = \int_S \nabla \cdot f(x) dx.$$  

**Example:** $\dot{x} = Ax, x \in \mathbb{R}^2$ can have periodic orbits only if $\text{Trace}(A) = 0$, e.g.,

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}.$$ 

**Example:**

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0 \\
\nabla \cdot f(x) &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta
\end{align*}$$ 

Therefore, no periodic orbit can lie entirely in the region $x_1 \leq -\sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$, or $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$ where $\nabla \cdot f(x) \leq 0$, or $x_1 \geq \sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$. 

![Diagram showing possible and not possible regions for periodic orbits](image-url)