EE222 - Spring'16 - Lecture 20 Notes[1]

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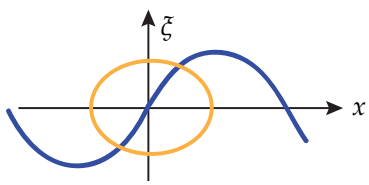
Feedback Linearization (continued)

Nonlinear Changes of Variables

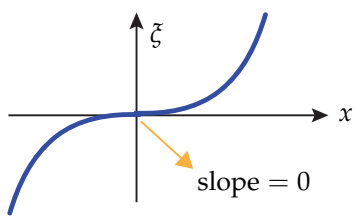
$T : \mathbb{R}^n \to \mathbb{R}^n$ is called a *diffeomorphism* if its inverse T^{-1} exists, and both T and T^{-1} are continuously differentiable (C^1).

Examples:

1. $\xi = Tx$ is a diffeomorphism if T is a nonsingular matrix
2. $\xi = \sin x$ is a local diffeomorphism around $x = 0$, but not global



3. $\xi = x^3$ is not a diffeomorphism because $T^{-1}(\cdot)$ is not C^1 at $\xi = 0$



How to check if $\xi = T(x)$ is a local diffeomorphism?

Implicit Function Theorem

Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1 and there exists $x_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^m$ such that

$$f(x_0, \xi_0) = 0.$$

If $\frac{\partial f}{\partial x}(x_0, \xi_0)$ is nonsingular, then in a neighborhood of (x_0, ξ_0),

$$f(x, \xi) = 0$$

has a unique solution $x = g(\xi)$ where g is C^1 at $\xi = \xi_0$.

Corollary: Let $f(x, \xi) = T(x) - \xi$. If $\frac{\partial T}{\partial x}$ is nonsingular at x_0, then $T(\cdot)$ is a local diffeomorphism around x_0.

A "Normal Form" that Explicitly Displays the Zero Dynamics

Theorem: If $\dot{x} = f(x) + g(x)u$, $y = h(x)$ has a well-defined relative degree $r \leq n$, then there exist a diffeomorphism $T : x \to \begin{bmatrix} z \\ \zeta \end{bmatrix}$, $z \in \mathbb{R}^{n-r}$, $\zeta \in \mathbb{R}^r$, that transforms the system to the form:

$$\boxed{\begin{aligned} \dot{z} &= f_0(z,\zeta) \\ \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_r &= b(z,\zeta) + a(z,\zeta)u, \quad y = \zeta_1. \end{aligned}} \tag{1}$$

In particular, $\dot{z} = f_0(z,0)$ represents the zero dynamics. □

To obtain this form, let $\zeta = [h(x) \quad L_f h(x) \quad \dots \quad L_f^{r-1}h(x)]^T$, and find $n-r$ independent variables z such that $\dot{z}$ does not contain u.

Note that the terms $b(z,\zeta)$ and $a(z,\zeta)$ correspond to $L_f^r(x)$ and $L_g L_f^{r-1}h(x)$ in the original coordinates.

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_3 + u \\ \dot{x}_3 &= \beta x_3 - u \\ y &= x_1. \end{aligned}$$

Let $\zeta_1 = x_1$, $\zeta_2 = x_2$, and note that $z = x_2 + x_3$ is independent of ζ_1, ζ_2, and $\dot{z}$ does not contain u. Thus, the normal form is:

$$\begin{aligned} \dot{z} &= (\alpha+\beta)x_3 = (\alpha+\beta)z - (\alpha+\beta)\zeta_2 \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \alpha x_3 + u = \alpha z - \alpha\zeta_2 + u. \end{aligned}$$

I/O Linearizing Controller in the new coordinates (1):

$$u = \frac{1}{a(z,\zeta)}\Big(-b(z,\zeta) + v\Big) \tag{2}$$

$$v = -k_1\zeta_1 \cdots - k_r\zeta_r \tag{3}$$

where $k_1, \cdots, k_r$ are such that all roots of $s^r + k_r s^{r-1} + \cdots + k_2 s + k_1$ have negative real parts.

Theorem: If $z = 0$ is locally exponentially stable for the zero dynamics $\dot{z} = f_0(z,0)$, then (2)–(3) locally exponentially stabilizes $x = 0$.

Proof: Closed-loop system:

$$\begin{aligned} \dot{z} &= f_0(z,\zeta) \\ \dot{\zeta} &= A\zeta \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & & \ddots & \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_r \end{bmatrix}$$

is Hurwitz. The Jacobian linearization at $(z, \zeta) = 0$ is:

$$J = \begin{bmatrix} \frac{\partial f_0}{\partial z}(0,0) & \frac{\partial f_0}{\partial \zeta}(0,0) \\ 0 & A \end{bmatrix}$$

where $\frac{\partial f_0}{\partial z}(0,0)$ is Hurwitz since $\dot{z} = f_0(z, 0)$ is exponentially stable by the proposition in Lecture 11, page 1. Since A is also Hurwitz, all eigenvalues of J have negative real parts $\Rightarrow$ exponential stability.

Global asymptotic stability can be guaranteed with additional assumptions on the zero dynamics, such as ISS of

$$\dot{z} = f_0(z, \zeta)$$

with respect to the input ξ:

$$\boxed{\dot{\zeta} = A\zeta} \xrightarrow{\zeta} \boxed{\dot{z} = f_0(z, \zeta)}$$

<u>Example:</u> $\dot{z} = -z + z^2\zeta, \quad \dot{\zeta} = -k\zeta$

Note: the z subsystem is not ISS

$(z, \zeta) = 0$ is locally exponentially stable, but not globally: solutions escape in finite time for large $z(0)$.

I/O Linearizing Controller for Tracking

For the output $y(t)$ to track a reference signal[2] $y_d(t)$, replace (3) with:

[2] assumed to be r times differentiable

$$v = -k_1(\zeta_1 - y_d(t)) - k_2(\zeta_2 - \dot{y}_d(t)) \cdots - k_r(\zeta_r - y_d^{(r-1)}(t)) + y_d^{(r)}(t)$$

Let $e_1 \triangleq \zeta_1 - y_d(t)$, $e_2 \triangleq \zeta_2 - \dot{y}_d(t)$, $\dots$, $e_r \triangleq \zeta_r - y_d^{(r-1)}(t)$. Then:

$$\left.\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_r &= v - y_d^{(r)}(t) = -k_1 e_1 - \cdots - k_r e_r \end{aligned}\right\} \quad \dot{e} = Ae.$$

Thus $e(t) \to 0$, that is $y(t) - y_d(t) \to 0$.

If $y_d(t)$ and its derivatives are bounded, then $\zeta(t)$ is bounded. If the zero dynamics $\dot{z} = f_0(z, \zeta)$ is ISS with respect to ζ, then $z(t)$ is also bounded. Thus, all internal signals are bounded.

Full-State Feedback Linearization

The system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, is (full state) feedback linearizable if a function $h(x)$ exists such that the relative degree from u to $y = h(x)$ is n.

Since $r = n$, the normal form (1) has no zero dynamics and

$$x \to \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

is a diffeomorphism that transforms the system to the form:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u. \end{aligned} \tag{4}$$

Then, (2)-(3) with $r = n$ is a feedback linearizing controller.

Closed-loop system in the new coordinates:

$$\dot{\zeta} = A\zeta \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & & \ddots & \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_r \end{bmatrix}.$$

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \end{aligned} \tag{5}$$

The choice $y = x_3$ gives relative degree $r = n = 3$.
Let $\zeta_1 = x_3$, $\zeta_2 = \dot{x}_3 = x_1 - x_3$, $\zeta_3 = \ddot{x}_3 = \dot{x}_1 - \dot{x}_3 = x_2 + 2x_1^2 - x_1 + x_3$.

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ \dot{\zeta}_3 &= \underbrace{(4x_1 - 1)(x_2 + 2x_1^2) + x_1}_{L_f^n h(x)} + u \end{aligned}$$

Feedback linearizing controller:

$$\begin{aligned} u &= -(4x_1 - 1)(x_2 + 2x_1^2) - x_1 - k_1\zeta_1 - k_2\zeta_2 - k_3\zeta_3 \\ &= -(4x_1 - 1)(x_2 + 2x_1^2) - x_1 - k_1 x_3 - k_2(x_1 - x_3) \\ &\qquad - k_3(x_2 + 2x_1^2 - x_1 + x_3). \end{aligned}$$