Feedback Linearization (continued)

Nonlinear Changes of Variables

$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a *diffeomorphism* if its inverse $T^{-1}$ exists, and both $T$ and $T^{-1}$ are continuously differentiable ($C^1$).

Examples:

1. $\xi =Tx$ is a diffeomorphism if $T$ is a nonsingular matrix
2. $\xi = \sin x$ is a local diffeomorphism around $x = 0$, but not global

![Diffeomorphism Example](image)

3. $\xi = x^3$ is not a diffeomorphism because $T^{-1}(\cdot)$ is not $C^1$ at $\xi = 0$

![Non-Diffeomorphism Example](image)

How to check if $\xi = T(x)$ is a local diffeomorphism?

Implicit Function Theorem

Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is $C^1$ and there exists $x_0 \in \mathbb{R}^n, \xi_0 \in \mathbb{R}^m$ such that

$$f(x_0, \xi_0) = 0.$$

If $\frac{\partial f}{\partial x}(x_0, \xi_0)$ is nonsingular, then in a neighborhood of $(x_0, \xi_0)$,

$$f(x, \xi) = 0$$

has a unique solution $x = g(\xi)$ where $g$ is $C^1$ at $\xi = \xi_0$.

Corollary: Let $f(x, \xi) = T(x) - \xi$. If $\frac{\partial T}{\partial x}$ is nonsingular at $x_0$, then $T(\cdot)$ is a local diffeomorphism around $x_0$. 
A "Normal Form" that Explicitly Displays the Zero Dynamics

Theorem: If $\dot{x} = f(x) + g(x)u, y = h(x)$ has a well-defined relative degree $r \leq n$, then there exist a diffeomorphism $T : x \rightarrow \begin{bmatrix} z \\ \zeta \end{bmatrix}$, $z \in \mathbb{R}^{n-r}, \zeta \in \mathbb{R}^r$, that transforms the system to the form:

\[
\begin{align*}
\dot{z} &= f_0(z, \zeta) \\
\dot{\zeta}_1 &= \zeta_2 \\
&\vdots \\
\dot{\zeta}_r &= b(z, \zeta) + a(z, \zeta)u, \quad y = \zeta_1.
\end{align*}
\]  

(1)

In particular, $\dot{z} = f_0(z, 0)$ represents the zero dynamics. □

To obtain this form, let $\zeta = [h(x) \quad L_fh(x) \ldots \quad L_{f}^{r-1}h(x)]^T$, and find $n - r$ independent variables $z$ such that $\dot{z}$ does not contain $u$.

Note that the terms $b(z, \zeta)$ and $a(z, \zeta)$ correspond to $L_r f(x)$ and $L_g L_{r-1}f(x)$ in the original coordinates.

Example:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= ax_3 + u \\
\dot{x}_3 &= \beta x_3 - u \\
y &= x_1.
\end{align*}
\]

Let $\zeta_1 = x_1, \zeta_2 = x_2$, and note that $z = x_2 + x_3$ is independent of $\zeta_1, \zeta_2$, and $\dot{z}$ does not contain $u$. Thus, the normal form is:

\[
\begin{align*}
\dot{z} &= (a + \beta)x_3 = (a + \beta)z - (a + \beta)\zeta_2 \\
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= ax_3 + u = az - a\zeta_2 + u.
\end{align*}
\]

I/O Linearizing Controller in the new coordinates (1):

\[
\begin{align*}
u &= \frac{1}{a(z, \zeta)} \left( - b(z, \zeta) + v \right) \\
v &= -k_1\zeta_1 \cdots - k_r\zeta_r
\end{align*}
\]

(2) (3)

where $k_1, \ldots, k_r$ are such that all roots of $s^r + k_1 s^{r-1} + \cdots + k_2 s + k_1$ have negative real parts.

Theorem: If $z = 0$ is locally exponentially stable for the zero dynamics $\dot{z} = f_0(z, 0)$, then (2)–(3) locally exponentially stabilizes $x = 0$.

Proof: Closed-loop system:

\[
\begin{align*}
\dot{z} &= f_0(z, \zeta) \\
\dot{\zeta} &= A\zeta
\end{align*}
\]
where

\[
A = \begin{bmatrix}
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
& & & \\
-k_1 & -k_2 & -k_3 & \ldots & -k_r
\end{bmatrix}
\]

is Hurwitz. The Jacobian linearization at \((z, \zeta) = 0\) is:

\[
J = \begin{bmatrix}
\frac{\partial f_0}{\partial z}(0, 0) & \frac{\partial f_0}{\partial \zeta}(0, 0) \\
0 & A
\end{bmatrix}
\]

where \(\frac{\partial f_0}{\partial z}(0, 0)\) is Hurwitz since \(\dot{z} = f_0(z, 0)\) is exponentially stable by the proposition in Lecture 11, page 1. Since \(A\) is also Hurwitz, all eigenvalues of \(J\) have negative real parts \(\Rightarrow\) exponential stability.

*Global* asymptotic stability can be guaranteed with additional assumptions on the zero dynamics, such as ISS of \(\dot{z} = f_0(z, \zeta)\) with respect to the input \(\xi\):

\[
\dot{z} = f_0(z, \zeta) \quad \dot{\zeta} = A\zeta
\]

**Example:** \(\dot{z} = -z + z^2\zeta, \quad \dot{\zeta} = -k\zeta\)

\((z, \zeta) = 0\) is locally exponentially stable, but not globally: solutions escape in finite time for large \(z(0)\).

**I/O Linearizing Controller for Tracking**

For the output \(y(t)\) to track a reference signal\(^2\) \(y_d(t)\), replace (3) with:

\[
v = -k_1(\zeta_1 - y_d(t)) - k_2(\zeta_2 - \dot{y}_d(t)) - \cdots - k_r(\zeta_r - y_d^{(r-1)}(t)) + y_d^{(r)}(t)
\]

Let \(e_1 \triangleq \zeta_1 - y_d(t), e_2 \triangleq \zeta_2 - \dot{y}_d(t), \ldots, e_r \triangleq \zeta_r - y_d^{(r-1)}(t)\). Then:

\[
\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
& \vdots \\
\dot{e}_r &= v - y_d^{(r)}(t) = -k_1e_1 - \cdots - k_re_r
\end{align*}
\]

Thus \(e(t) \to 0\), that is \(y(t) - y_d(t) \to 0\).

If \(y_d(t)\) and its derivatives are bounded, then \(\zeta(t)\) is bounded. If the zero dynamics \(\dot{z} = f_0(z, \zeta)\) is ISS with respect to \(\zeta\), then \(z(t)\) is also bounded. Thus, all internal signals are bounded.
**Full-State Feedback Linearization**

The system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, is (full state) feedback linearizable if a function $h(x)$ exists such that the relative degree from $u$ to $y = h(x)$ is $n$.

Since $r = n$, the normal form (1) has no zero dynamics and

$$x \rightarrow \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{bmatrix} = \begin{bmatrix} h(x) \\ L_fh(x) \\ \vdots \\ L^n_fh(x) \end{bmatrix}$$

is a diffeomorphism that transforms the system to the form:

$$\begin{align*}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \zeta_3 \\
&\vdots \\
\dot{\zeta}_n &= L^n_fh(x) + L_{g}L^{n-1}_f h(x)u.
\end{align*}$$

Then, (2)-(3) with $r = n$ is a feedback linearizing controller.

Closed-loop system in the new coordinates:

$$\dot{\xi} = A\xi \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & \ddots & \vdots \\ -k_1 & -k_2 & -k_3 & \cdots & -k_r \end{bmatrix}.$$

Example:

$$\begin{align*}
\dot{x}_1 &= x_2 + 2x_1^2 \\
\dot{x}_2 &= x_3 + u \\
\dot{x}_3 &= x_1 - x_3
\end{align*}$$

The choice $y = x_3$ gives relative degree $r = n = 3$.

Let $\zeta_1 = x_3$, $\zeta_2 = x_3 = x_1 - x_3$, $\zeta_3 = \ddot{x}_3 = \dot{x}_1 - \dot{x}_3 = x_2 + 2x_1^2 - x_1 + x_3$.

$$\begin{align*}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \zeta_3 \\
\dot{\zeta}_3 &= (4x_1 - 1)(x_2 + 2x_1^2) + x_1 + u
\end{align*}$$

Feedback linearizing controller:

$$\begin{align*}
u &= -(4x_1 - 1)(x_2 + 2x_1^2) - x_1 - k_1\zeta_1 - k_2\zeta_2 - k_3\zeta_3 \\
&= -(4x_1 - 1)(x_2 + 2x_1^2) - x_1 - k_1x_3 - k_2(x_1 - x_3) \\
&\quad - k_3(x_2 + 2x_1^2 - x_1 + x_3).
\end{align*}$$