EE222 - Spring'16 - Lecture 22 Notes[1]

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Dissipativity Theory

The notion of *dissipativity*[2] characterizes dynamical systems by how their inputs and outputs correlate. This correlation is described by a scalar valued *supply rate* $s(u,y)$ the choice of which distinguishes the type of dissipativity.

[2] J.C. Willems, "Dissipative Dynamical Systems, Part I: General theory; Part II: Linear systems with quadratic supply rates", *Archive for Rational Mechanics and Analysis*, pages 321-393, 1972.

<u>Definition:</u> The system below, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$,

$$\dot{x} = f(x,u) \qquad f(0,0) = 0 \tag{1}$$

$$y = h(x,u) \qquad h(0,0) = 0, \tag{2}$$

is said to be *dissipative* with respect to a *supply rate* $s(u,y)$ if there exists a C^1 function $V : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ such that $V(0) = 0$ and

$$\nabla V(x)^T f(x,u) \leq s(u, h(x,u)) \qquad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m. \tag{3}$$

V is called a *storage function*.

Noting that the left hand side of (3) equals $\frac{d}{dt}V(x(t))$, and integrating from $t = 0$ to τ, we get

$$V(x(\tau)) - V(x(0)) \leq \int_0^\tau s(u(t), y(t))dt. \tag{4}$$

Since $V(x(\tau)) \geq 0$, (4) implies

$$-V(x(0)) \leq \int_0^\tau s(u(t), y(t))dt. \tag{5}$$

Thus, the integral of the supply rate $s(u(t), y(t))$ along the trajectories is nonnegative when $x(0) = 0$. This means that the system favors a positive sign for $s(u(t), y(t))$ when averaged over time.

Important special cases of dissipativity are discussed below.

- **Finite L_2-gain:** $\quad s(u,y) = \gamma^2|u|^2 - |y|^2 \quad \gamma > 0$

The L_2 norm of a signal $u(t)$ is defined as

$$\lim_{\tau \to \infty} \sqrt{\int_0^\tau |u(t)|^2 dt}$$

when the limit exists. Note from (5) that

$$-V(x(0)) \leq \gamma^2 \int_0^\tau |u(t)|^2 dt - \int_0^\tau |y(t)|^2 dt$$

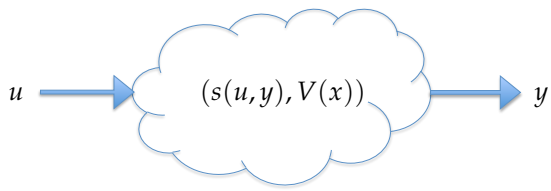

Figure 1: Dissipativity characterizes a dynamical system with a *supply rate* $s(u,y)$ that describes how the inputs and outputs correlate, and an accompanying *storage function* V.

$$\Rightarrow \quad \int_0^\tau |y(t)|^2 dt \leq \gamma^2 \int_0^\tau |u(t)|^2 dt + V(x(0)).$$

Taking square roots of both sides and applying the inequality $\sqrt{a^2+b^2} \leq |a|+|b|$ to the right-hand side we get

$$\sqrt{\int_0^\tau |y(t)|^2 dt} \leq \gamma \sqrt{\int_0^\tau |u(t)|^2 dt} + \sqrt{V(x(0))}.$$

This means that the L_2 norm of $y(t)$ is bounded by that of $u(t)$ multiplied by γ, plus an offset term due to initial conditions. Thus γ serves as an L_2 gain for the system.

- **Passivity:** $\quad s(u,y) = u^T y$

With this choice of supply rate, (5) implies

$$\int_0^\tau u(t)^T y(t) dt \geq -V(x(0)) \tag{6}$$

which favors a positive sign for the inner product of $u(t)$ and $y(t)$. Periods of time when $u(t)^T y(t) < 0$ must be outweighed by those when $u(t)^T y(t) > 0$.

- **Output Strict Passivity:** $\quad s(u,y) = u^T y - \varepsilon |y|^2 \quad \varepsilon > 0$

This supply rate implies passivity since $s(u,y) \leq u^T y$, but is more stringent than (6):

$$\int_0^\tau u(t)^T y(t) dt \geq -V(x(0)) + \underbrace{\varepsilon \int_0^\tau |y(t)|^2 dt}_{\geq 0}.$$

Output strict passivity also implies an L_2 gain of $\gamma = 1/\varepsilon$ because a completion of squares argument gives

$$u^T y - \frac{1}{\gamma} y^T y \leq \frac{\gamma}{2} u^T u - \frac{1}{2\gamma} y^T y \tag{7}$$

where the right-hand side is equal to

$$\frac{1}{2\gamma}(\gamma^2 |u|^2 - |y|^2).$$

Thus we use $2\gamma V$ as a storage function and conclude dissipativity with the L_2 gain supply rate $\gamma^2 |u|^2 - |y|^2$.

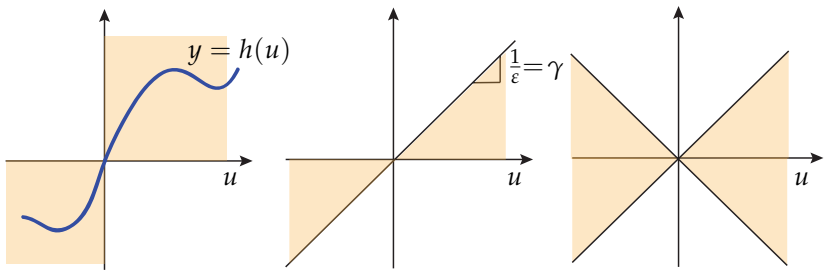

Figure 2: The graph of a passive static nonlinearity $h(\cdot)$ must lie in the first and third quadrants (left). Output strict passivity confines $h(\cdot)$ to the narrower sector (middle) and a gain bound γ corresponds to the sector upper and lower bounded by the lines $\pm\gamma u$ (right).

Graphical Interpretation

For a memoryless system

$$y(t) = h(u(t))$$

we take the storage function in (3) to be zero and interpret dissipativity as the static inequality

$$s(u, h(u)) \geq 0 \qquad \forall u \in \mathbb{R}^m. \tag{8}$$

This inequality characterizes a class of maps $h(\cdot)$ that are dissipative with respect to the supply rate $s(\cdot,\cdot)$. For example, a scalar function $h(\cdot)$ is passive if $uh(u) \geq 0$ for all u, which means that the graph of $h(\cdot)$ lies in the first and third quadrants as in Figure 2 (left). Likewise, the sector in the middle represents the output strict passivity supply rate $s(u, y) = uy - \varepsilon y^2, \varepsilon > 0$, and the sector on the right represents the finite gain supply rate $s(u, y) = \gamma^2 u^2 - y^2$.

Examples of Passive Systems

Suppose we wish to prove passivity of the system

$$\begin{aligned} \dot{x} &= f_0(x) + g(x)u \\ y &= h(x) \end{aligned}$$

which is a special case of (1)-(2) with $f(x, u) = f_0(x) + g(x)u$ affine in u and $h(x, u) = h(x)$ independent of u. Then (3) becomes

$$\nabla V(x)^T f_0(x) + \nabla V(x)^T g(x)u \leq h(x)^T u \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m \tag{9}$$

which is equivalent to

$$\nabla V(x)^T f_0(x) \leq 0 \qquad \nabla V(x)^T g(x) = h^T(x) \qquad \forall x \in \mathbb{R}^n. \tag{10}$$

The inequality in (10) follows by setting $u = 0$ in (9). To see how the equality follows suppose there exists an x for which $\nabla V(x)^T g(x) -$

$h^T(x) \neq 0$. Then we can select a u such that $(\nabla V(x)^T g(x) - h^T(x))u$ is positive and large enough to contradict (9).

Similar arguments show that output strict passivity is equivalent to

$$\nabla V(x)^T f_0(x) \leq -\varepsilon h(x)^T h(x) \;\; \nabla V(x)^T g(x) = h^T(x) \;\; \forall x \in \mathbb{R}^n. \quad (11)$$

Example 1: Consider the scalar system

$$\dot{x} = f_0(x) + u, \quad y = h(x), \quad u, x, y \in \mathbb{R} \quad (12)$$

where $xh(x) \geq 0$ for all x, as in Figure 2 (left). For this system the equality in (11) is

$$\frac{dV(x)}{dx} = h(x)$$

whose solution subject to $V(0) = 0$ is

$$V(x) = \int_0^x h(z)dz. \quad (13)$$

Furthermore $V(x) \geq 0$ because $h(z)$ and dz have equal signs (positive when the limit of integration is $x > 0$ and negative when $x < 0$).

The inequality condition in (11) is then

$$h(x)(f_0(x) + \varepsilon h(x)) \leq 0$$

which is equivalent to

$$x(f_0(x) + \varepsilon h(x)) \leq 0 \quad (14)$$

since $xh(x) \geq 0$. Thus, we conclude passivity when (14) holds with $\varepsilon = 0$ and output strict passivity when (14) holds with $\varepsilon > 0$.

Note that an integrator, where $f_0(x) \equiv 0$, is passive since (14) holds with $\varepsilon = 0$, but not output strictly passive since (14) with $\varepsilon > 0$ contradicts the assumption $xh(x) \geq 0$.

Example 2: Consider the second order model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_2 - \phi'(x_1) + u \\ y &= x_2 \end{aligned}$$

where $\phi'(\cdot)$ is the derivative of a continuously differentiable and nonnegative function $\phi(\cdot)$ satisfying $\phi(0) = 0$. We interpret x_1 as position, x_2 as velocity, u as force, $k \geq 0$ as damping coefficient, and $\phi(x_1)$ as potential energy of a mechanical system.

For this system the equality condition $\nabla V(x)^T g(x) = h^T(x)$ becomes:

$$\frac{\partial V(x_1, x_2)}{\partial x_2} = x_2.$$

Thus we restrict the storage function to be of the form:

$$V(x_1, x_2) = V_1(x_1) + \frac{1}{2}x_2^2$$

and examine the inequality condition $\nabla V(x)^T f_0(x) \leq 0$. We have

$$\begin{aligned} \nabla V(x)^T f_0(x) &= \frac{dV_1(x_1)}{dx_1}x_2 + x_2\left(-kx_2 - \phi'(x_1)\right) \\ &= -kx_2^2 + x_2\left(\frac{dV_1(x_1)}{dx_1} - \phi'(x_1)\right). \end{aligned}$$

The choice $V_1(x_1) = \phi(x_1)$ ensures $\nabla V(x)^T f_0(x) = -kx_2^2 = -kh(x)^2$ which proves passivity when $k = 0$ and output strict passivity when $k > 0$.

The resulting storage function $V(x_1, x_2) = \phi(x_1) + \frac{1}{2}x_2^2$ is the sum of potential and kinetic energy terms, and $u(t)y(t)$ =force·velocity may be interpreted as the power supplied to the system. The definition of dissipativity (4) is thus consistent with the physical notion of energy storage, and dissipation when damping is present.