Case Study: A Vehicle Platoon

Consider a platoon where the velocity of each vehicle is governed by

\[ \dot{v}_i = -v_i + v_i^0 + u_i \quad i = 1, \ldots, N \]

(1)

in which \( u_i \) is a coordination feedback to be designed and \( v_i^0 \) is the (constant) nominal velocity of vehicle \( i \) in the absence of such feedback. The position of vehicle \( i \) is then obtained from

\[ \dot{x}_i = v_i. \]

We will design feedback laws that depend on relative positions with respect to a subset of other vehicles, typically nearest neighbors.

We introduce an undirected graph where the vertices represent the vehicles and an edge between vertices \( i \) and \( j \) means that vehicles \( i \) and \( j \) have access to the relative position measurement \( x_i - x_j \). Next we assign an orientation to each edge by selecting one end to be the head and the other to be the tail. Then the *incidence matrix*

\[ D_{il} = \begin{cases} 
1 & \text{if vertex } i \text{ is the head of edge } l \\
-1 & \text{if vertex } i \text{ is the tail of edge } l \\
0 & \text{otherwise} 
\end{cases} \]

(2)

generates a vector of relative positions \( z_l \) for the edges \( l = 1, \ldots, L \) by

\[ z = D^T x. \]

(3)

As an illustration, in Figure 1,

\[ D = \begin{bmatrix} 
1 & 0 \\
-1 & 1 \\
0 & -1 
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_1 \\
z_2 
\end{bmatrix} = D^T x = \begin{bmatrix} x_1 - x_2 \\
x_2 - x_3 
\end{bmatrix}. \]

We propose the feedback law

\[ u = -D \begin{bmatrix} h_1(z_1) \\
\vdots \\
h_L(z_L) 
\end{bmatrix} \]

(4)
where each function \( h_l : \mathbb{R} \rightarrow \mathbb{R} \) is increasing and onto. This means that vehicle \( i \) applies the input
\[
u_i = - \sum_{l=1}^{L} D_{il} h_l(z_l)
\]which depends on locally available measurements because \( D_{il} \neq 0 \) when vertex \( i \) is the head or tail of edge \( l \). In the case of Figure 1,
\[
\begin{align*}
u_1 &= -h_1(z_1) & u_2 &= h_1(z_1) - h_2(z_2) & u_3 &= h_2(z_2)
\end{align*}
\]where we may interpret \( h_1(z_1) \) and \( h_2(z_2) \) as virtual spring forces between vehicles 1 and 2, and 2 and 3 respectively.

Note from (3) that
\[
\dot{z} = D^Tv \triangleq w
\]where we interpret \( w \) as an input and define the output
\[
y \triangleq \begin{bmatrix} h_1(z_1) \\ \vdots \\ h_L(z_L) \end{bmatrix}.
\]Then the closed-loop system is as in Figure 2 (left) where the feed-forward blocks \( u_i \mapsto v_i \) represent the velocity dynamics (1) and the feedback blocks \( w_l \mapsto y_l \) represent the \( l \)th subsystem of the relative position dynamics (6)-(7).

This block diagram is equivalent to the one in Figure 2 (right) which is of the standard form in Lecture 23 with the interconnection matrix
\[
M = \begin{bmatrix} 0 & -D \\ D^T & 0 \end{bmatrix}.
\]The skew symmetry of \( M \) will allow us to conclude stability from the passivity properties of the subsystems.
Determining the Equilibrium

At equilibrium the right hand side of (6) must vanish, that is

\[ D^T v^* = 0. \]  \hspace{1cm} (9)

By the definition (2) above, the null space of \( D^T \) includes the vector of ones: \( D^T \mathbb{1} = 0 \). In addition, if the graph is connected then the span of \( \mathbb{1} \) constitutes the entire null space: there is no solution to (9) other than \( v^* = \alpha \mathbb{1} \) where \( \alpha \) is a common platoon velocity.

Setting the right hand side of (1) to zero, we see that the equilibrium value of the inputs \( u_i \) must compensate for the variations in the nominal velocities \( v^*_i \) so that a common velocity \( \alpha \) can be maintained:

\[ -\alpha + v^*_i + u_i = 0 \quad i = 1, \ldots, N. \]  \hspace{1cm} (10)

Note that \( \sum_{i=1}^{N} u_i = \mathbb{1}^T u = 0 \), which follows from (4) and \( \mathbb{1}^T D = 0 \).

Thus, if we add the equation (10) for \( i = 1 \) to \( i = N \) we get

\[ -N\alpha + \sum_{i=1}^{N} v^*_i = 0 \]

which shows that the common velocity \( \alpha \) must be the average \( \frac{1}{N} \sum_{i=1}^{N} v^*_i \).

Substituting this average for \( \alpha \) and (5) for \( u^*_i \) back in (10) we obtain the following equations for \( z^*_l \):

\[ v^*_i - \frac{1}{N} \sum_{i=1}^{N} v^*_i = \sum_{l=1}^{L} D_{il} h_l(z^*_i) \quad i = 1, \ldots, N. \]

These equations are particularly transparent for a line graph as in Figure 1 where the head and tail of edge \( l \) are vertices \( l \) and \( l+1 \):

\[ v^*_1 - \frac{1}{N} \sum_{i=1}^{N} v^*_i = h_1(z^*_1) \]
\[ v^*_i - \frac{1}{N} \sum_{i=1}^{N} v^*_i = -h_{i-1}(z^*_{i-1}) + h_i(z^*_i) \quad i = 2, \ldots, N - 1 \]
\[ v^*_N - \frac{1}{N} \sum_{i=1}^{N} v^*_i = -h_{N-1}(z^*_{N-1}). \]

Adding equations \( i = 1 \) to \( l \) we get a new equation that depends only on \( h_l(z^*_l) \). Then a solution \( z^*_l \) exists since \( h_l(\cdot) \) is onto, and is unique since \( h_l(\cdot) \) is increasing. A similar argument may be developed for other acyclic graphs.
**Stability Analysis**

To analyze the stability of the equilibrium characterized above, we define the shifted state variables

\[ \tilde{v}_i \triangleq v_i - \alpha \quad \tilde{z}_l \triangleq z_l - z_l^* \]

so that, at equilibrium \( \tilde{v}_i = 0 \) and \( \tilde{z}_l = 0 \).

From (1) and (10),

\[ \dot{\tilde{v}}_i = -v_i + v_i^0 + u_i = -\tilde{v}_i + \tilde{u}_i \quad (11) \]

which is output strictly passive with input \( \tilde{u}_i \triangleq u_i - u_i^* \) and output \( \tilde{v}_i \), since the storage function

\[ V_i(\tilde{v}_i) = \frac{1}{2} \tilde{v}_i^2 \]

satisfies

\[ \dot{V}_i = -\tilde{v}_i^2 + \tilde{v}_i \tilde{u}_i. \]

Likewise, from (6)-(7),

\[ \dot{\tilde{z}}_l = w_l \quad (12) \]

which is passive with input \( w_l \) and output:

\[ \tilde{y}_l \triangleq h_l(z_l) - h_l(z_l^*). \]

To see this, take the storage function

\[ W_l(\tilde{z}_l) = \int_{z_l^*}^{z_l} [h_l(z_l^* + \sigma) - h_l(z_l^*)] d\sigma \]

which satisfies

\[ \dot{W}_l = [h_l(z_l) - h_l(z_l^*)] w_l = \tilde{y}_l w_l. \]

It follows from the skew symmetry of the interconnection matrix \( M \) and the passivity of the subsystems (see Lecture 24) that the origin \( \tilde{v}_i = 0 \) and \( \tilde{z}_l = 0 \) is stable and a Lyapunov function is

\[ \sum_{i=1}^{N} V_i(\tilde{v}_i) + \sum_{l=1}^{L} W_l(\tilde{z}_l). \]