Linear Systems

Consider the system

\[ \dot{x} = Ax, \quad x \in \mathbb{R}^2, \quad A \in \mathbb{R}^{2 \times 2}. \]  

(1)

Characterize (1) for the following cases:

\[ A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \]

where \( a, b, \alpha, \beta \in \mathbb{R} \). What are the eigenvalues? Sketch the systems in the plane.

Now consider

\[ \dot{x} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x \]  

(2)

Characterize (2). Sketch its dynamics.

Hartman-Grobman Theorem

**Theorem 1** (Informal). The nonlinear system \( \dot{x} = f(x) \) has the same qualitative structure as the linear system \( \dot{x} = Ax \) with \( A = Df(x^*) \), where \( x^* \) is an equilibrium of the nonlinear system, if \( A \) has no eigenvalue with zero real part.

Bendixson’s Theorem

**Theorem 2.** Suppose \( D \subset \mathbb{R}^2 \) is a simply connected region. Given \( \dot{x} = f(x), \; x \in \mathbb{R}^2 \), and \( \text{div} f \neq 0 \) and does not change signs in \( D \), then \( D \) contains no closed orbits of \( \dot{x} = f(x) \).

Duffing’s Equation

Consider

\[ \ddot{x} + \delta \dot{x} - x + x^3 = 0 \]  

(3)

Using Bendixson’s Theorem, what statements can you make with regard to closed orbits of (3)?
Poincaré-Bendixson Theorem

**Theorem 3.** Consider the planar dynamical system

\[
\dot{x}_1 = f_1(x_1, x_2) \\
\dot{x}_2 = f_2(x_1, x_2)
\]

Every compact (non-empty) positively invariant set \( M \) contains an equilibrium point or a closed orbit.

Pendulum

Consider the system \( \ddot{\theta} + \sin(\theta) = 0, \ \theta \in [-\pi, \pi) \). Find a first integral and sketch the system in the plane.