Problem 1

Let \((C([0, 1]), \mathbb{C})\) be the vector space of continuous functions from domain \([0, 1]\) to \(\mathbb{C}\) over the field \(\mathbb{C}\). Show that \(\langle \cdot, \cdot \rangle : C([0, 1]) \times C([0, 1]) \rightarrow \mathbb{C}\) with

\[
\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt
\]

defines an inner product on \(C([0, 1])\).

Problem 2

Let \(X\) be the space of real-valued, continuous functions defined on \([0, 1]\) with the following norm:

\[
\|x\|_X = \int_0^1 |x(t)| \, dt
\]

Is this a valid norm?

Problem 3

Show that the sequence \(\{\frac{1}{n}\}_{n=1}^\infty\) is a Cauchy Sequence.

Problem 4

Show that any convergent sequence is also a Cauchy Sequence.

Problem 5

Norm Equivalence. Show that for all \(x \in \mathbb{R}^n\)

\[
\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1.
\]

Problem 6

Consider an inner product space \(V\) with \(x, y \in V\). Using properties of the inner product, show that

\[
\|x + y\| + \|x - y\| = 2\|x\|^2 + 2\|y\|^2.
\]