Goals of this lecture:

- present Lyapunov theory for asymptotic and exponential stability
- examples of Lyapunov theory
- LaSalle's Theorem
- examples of LaSalle

Refs: Sastry § 5.3, 5.4, 5.5
**DEFN** $x_e = 0$ is said to be an **ASYMPTOTICALLY STABLE** equilibrium point of (NL) $[\dot{x} = f(x(t), t)]$ if:

1. $x_e = 0$ is **STABLE** (equilibrium point)
2. There exists a $S > 0$ such that $\|x_0\| < S \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0$

**Remark**. need both (1) and (2) . (2) does not imply (1).

**Example**:

\[ \begin{align*}
\dot{x}_1 &= x_1^2 - x_2^2 \\
\dot{x}_2 &= 2x_1x_2
\end{align*} \]

Identify $x_1 = \infty$ with $x_1 = -\infty$.

This system is not asymptotically stable because it is not even stable. Given $\epsilon > 0$, there are always initial conditions close to $x_e = 0$ which will exit the $\epsilon$-ball before converging to 0!
**Def.** $x_e = 0$ is a **globally asymptotically stable** equilibrium point of (NL) if

1. $x_e = 0$ is an asymptotically stable eq.
2. $\lim_{t \to \infty} x(t) = 0$ for all $x_0 \in \mathbb{R}^n$.

*(Note: Global stability is defined in the obvious way.)*

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**Def.** $x_e = 0$ is an **expontentially stable** equilibrium point of (NL) if there exists $m, \alpha > 0$ such that

$$
\|x(t)\| \leq m e^{-\alpha t} \|x_0\|
$$

for all $x_0 \in B_r$, $t > 0$. The constant $\alpha$ is an estimate of (and is called) the **rate of convergence**.

*(Note: Global exponential stability requires $x_0 \in \mathbb{R}^n$).*

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Exponential Stability $\subset$ Asymptotic Stability $\subset$ Stability
**NOTE 1:** The type of stability we are interested in depends on the engineering system at hand. For example, for a thermostat in a room, we are usually satisfied with mere stability, whereas for a car's cruise control system, or an aircraft's autopilot system, we require exponential stability.

**NOTE 2:** Even if the system (NC) is stable (asymptotic, exponential...) the solution $x(t)$ need not be continuous anywhere except at $x_e = 0$. 
LYAPUNOV ASYMPTOTIC STABILITY THEOREM

Consider \( \dot{x}(t) = f(x(t), t) \); \( x(t_0) = x_0 \) with equilibrium state \( x_e = 0 \).

If \( f > 0 \) such that
1. \( V(x,t) = p.d., \) decreasent, \( L-fn \) on \( G_r \)
2. \(-\dot{V}(x,t) = p.d \) on \( G_r \) [new condition]

Then \( x_e = \) asymptotically stable.

Intuition [not a formal proof].
1. \( \Rightarrow x_e \) is stable (1st) by Lyapunov stability Theorem.
2. \( \Rightarrow \left[ \frac{dV}{dt}(x(t), t) = \dot{V} < 0 \text{ whenever } x(t) \neq 0 \quad \forall t \geq t_0 \right] \)
   \( \Rightarrow [V(x(t), t) \to 0] \implies [x(t) \to 0] \)
   \( \uparrow_{V = p.d., decr.} \)
   \( \Rightarrow x_e = \) asymptotically stable (a.s)

NOTE: Global theorems result from replacing \( G_r \) with \( \mathbb{R}^n \).
### Summary (Basic Lyapunov Theorems)

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Sometimes use the following notation:

- **LPDF**: "locally PD function"
- **PDF**: "PD function"
Examples:

(1). \[
\begin{align*}
\dot{x}_1 &= -x_2 + x_1 \left(x_1^2 + x_2^2 - 1\right) \\
\dot{x}_2 &= x_1 + x_2 \left(x_1^2 + x_2^2 - 1\right)
\end{align*}
\]

Choose \[\nabla(x) = x_1^2 + x_2^2\] LPDF
(by inspection)
\[
\dot{\nabla}(x) = 2 \left(x_1^2 + x_2^2\right)(x_1^2 + x_2^2 - 1)
\]
So \[-\dot{\nabla}(x)\] is LPDF for
\[
\{x : x_1^2 + x_2^2 < 1\}
\]
\[\Rightarrow 0\] is a locally asymptotically stable equilibrium
(we knew this already by using this example before)

[Note: 0 is not globally asymptotically stable (abbr. G.A.S.) since there is a limit cycle of radius 1]
EXAMPLES

(2) \[\begin{array}{c}
x_1 \quad \text{charge on capacitor} \\
x_2 \quad \text{velocity through inductor}
\end{array}\]

\[f(x_2), g(x_1) \] \text{voltages.}

Resistor : Capacitor are locally Passive:
\[\begin{align*}
x f(x) &\geq 0 \quad \forall x \in [-x_0, x_0] \\
x g(x) &> 0 \quad \text{local}
\end{align*}\]

- use total energy of the system as a Lyapunov function candidate
\[V(x) = \frac{x_2^2}{2} + \int_{0}^{x_1} g(\xi) d\xi\]

\[V(x) \text{ is } \begin{cases} \text{LPDF} & \text{provided that } g(x_1) \text{ is not} \\
\text{DECR.} & \text{identically zero on some interval} \end{cases}\]

\[\dot{V}(x) = x_2 [-f(x_2) - g(x_1)] + g(x_1) x_2 \]
\[= -x_2 f(x_2) \leq 0\]

\[\Rightarrow \text{STABILITY} \quad \text{[of } (0,0) \text{]}\]
SYNCHRONOUS GENERATOR:

EXAMPLES:

\[
\Theta \quad \text{angle of rotor of generator}
\]

\[
(3) \quad \omega \quad \text{supply angle 0°}
\]

\[
P_m \quad \text{mechanical power input}
\]

\[
P_e \quad \text{electrical power output}
\]

\[
M \quad \text{moment of inertia of generator}
\]

\[
D \quad \text{generator's damping}
\]

\[
B \quad \text{susceptance of bus}
\]

\[
\dot{\omega} = \omega
\]

\[
\ddot{\omega} = - M^{-1} (D \omega + P_m - B \sin \Theta)
\]

\[
= - M^{-1} D \omega - M^{-1} \left( P_m - B \sin \Theta \right)
\]

\[
f(\omega)
\]

\[
g(\Theta)
\]

Equilibrium point at

\[
\omega_0 = 0
\]

\[
\Theta_0 = \sin^{-1} \frac{B}{P_m}
\]

Use same Lyapunov fn as before:

\[
V(\Theta, \omega) = \frac{1}{2} M \omega^2 + P_m \Theta + B \cos \Theta
\]

Translated: \( V(\Theta, \omega) - V(\Theta_0, \omega_0) \) is an LPDF around \((\Theta_0, \omega_0)\)

\[
\dot{V}(\Theta, \omega) = -D \omega^2 \leq 0
\]

\[\Rightarrow \text{STABILITY of (} \Theta_0, \omega_0 \text{)} \]
**CASALLE'S INVARIANCE PRINCIPLE.**

**[FOR TIME INVARIANT SYSTEMS]**

From the Lyapunov Stability Theorem, we know that if $v(x)$ is LPD (LPD) and $\dot{v}(x) \leq 0$, for $x \in \mathbb{R}^n$ or $x \in \mathbb{R}^n$.

Then $\dot{x} = f(x)$ is stable at 0 (globally stable).

However, we may still be able to prove asymptotic stability in this case using:

**CASALLE'S PRINCIPLE:**

Define $\Omega_c := \{ x \in \mathbb{R}^n : v(x) < c \}$

Suppose $\Omega_c$ is bounded, and $\dot{v} \leq 0$ for all $x \in \Omega_c$.

Define $S := \{ x \in \Omega_c : \dot{v}(x) = 0 \}$

Let $M$ be the largest invariant set in $S$. Then, whenever $x_0 \in \Omega_c$, $x(t)$ approaches $M$ as $t \to \infty$. \[\Box\]
$\tilde{u} \leq 0$

$S_c = \{ x \in \mathbb{R}^n : u(x) < c \}$

$S = \{ x \in S_c : u(x) = 0 \}$

$M$ is the largest invariant set in $S$.

**Idea of Proof:** if $x_0 \in S_c$, $x(t) \in S_c \forall t$.

Let $C_0 = \lim_{t \to \infty} u(x(t))$ (we know this exists since $u(x(t))$ is bounded below).

**Fact:** if a trajectory is completely enclosed within a bounded set, then the set of "limit points" that the trajectory can tend to (i.e. equilibria, limit cycles) is bounded. Further, the trajectory approaches this limit set as $t \to \infty$. [$Wiggins$ pp 46-50]

Let $L$ be the "limit set" of $x(t)$. Then $u(y) = C_0$ for $y \in L$, and $u(y) = 0$ for $y \not\in L$. Therefore $L \subseteq S$. But $L \subseteq M$ also since $L$ is invariant. $\therefore x(t) \to M \text{ as } t \to \infty$. 
How to use Casale's Principle to establish asymptotic stability?

Casale's Theorem (1960)

Given $\dot{x} = f(x)$, $\nu: \mathbb{R}^n \to \mathbb{R}$, l.p.d.f.

$\nu(x) \leq 0$ for $x \in \Omega_C$

where $\Omega_C = \{ x \in \mathbb{R}^n : \nu(x) \leq c \}$

Let $S = \{ x \in \Omega_C : \nu(x) = 0 \}$ which stays

Then if the only trajectory in $S$ is $x(t) = 0$, $\dot{x} = f(x)$ is locally asymptotically stable near $x^* = 0$. $

Remarks:

1) Global version of above:

(uses $\nu(x): \mathbb{R}^n \to \mathbb{R}^{PDF}$

$\nu(x) \leq 0$, $x \in \mathbb{R}^n$

$\Rightarrow$ Global Asymptotic Stability)

2) The theorem works because, even though $\nu(x) \leq 0$, the only trajectory of $\dot{x} = f(x)$ with $\nu(x(t)) = 0$ is $x(t) = 0$ (trivial trajectory).
EXAMPLES

1) \[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -f(x_2) - g(x_1)
\end{align*} \]

- Resistor & Capacitor are locally Passive
  \[ x f(x) \geq 0, \quad x g(x) \geq 0 \quad \forall x \in [-x_0, x_0] \]

- As before \[ V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\xi) d\xi \]
  \[ V(x) \text{ is LPDF } [\text{Assuming no deadband in } g(\cdot)] \]
  \[ \dot{V}(x) = -x_2 f(x_2) \leq 0 \]
  \[ \Rightarrow \text{STABILITY of } (0,0) \]

CAN WE SAY MORE?

Let \[ D := \{ x \in \mathbb{R}^2 / -x_0 < x_i < x_0 \}, \quad i = 1, 2 \]
\[ S = \{ x \in D / V(x) = 0 \} \]

To characterize \( S \), note that
\[ \dot{V}(x) = 0 \Rightarrow x_2 f(x_2) = 0 \]
\[ \Rightarrow x_2 = 0, \quad \text{since } -x_0 < x_2 < x_0. \]
EXAMPLES

1) (cont'd)

\[ S = \{ x \in D \mid x_2 = 0 \} \]

Now suppose \( x(t) \) is a trajectory that stays in \( S \):

\[ x_2 = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow g(x_1(t)) = 0 \Rightarrow x_1(t) = 0 \]

Therefore, the only solution that can stay in \( S \) is \( x(t) = 0 \).

Therefore \( x^* = 0 \) is asymptotically stable.
EXAMPLES

2). Consider the system

\[ \dot{y} = ay + u \quad y \in \mathbb{R}, u \in \mathbb{R} \]

\( u \) is called the "control" and we can manipulate it to make the system do what we want.

Consider the adaptive control law

\[ u = -ky, \quad k = \gamma y^2, \quad \gamma > 0 \]

The closed loop system may be written as:

\[ \dot{y} = -(k-a)y \]

\[ k = \gamma y^2 \]

\( y = 0 \) is a line of equilibria.

We want to show that the trajectory of the system approaches this equilibrium set as \( t \to \infty \), meaning that the adaptive controller succeeds in regulating \( y \) to zero.
EXAMPLES (2, cont'd).

Consider the Lyapunov function candidate

\[ v(y, k) = \frac{1}{2} y^2 + \frac{1}{2} \gamma (k-b)^2 \]

where \( b > a \).

\[ \dot{v}(y, k) = -y^2 (k-a) + y^2 (k-b) = -y^2 (b-a) \leq 0. \]

For any finite \( c > 0 \), the set

\[ \Omega_c = \{ [y] \in \mathbb{R}^2 \mid v(y, k) \leq c \} \]

is positively invariant bounded.

\[ S = \{ [y] \in \Omega_c \mid y = 0 \} \]

\[ M = S. \]

From Lasalle's Theorem, every trajectory starting in \( \Omega_c \), \([y](t)\), approaches \( M \) as \( t \to \infty \) [this actually holds globally]

\[ \therefore y(t) \to 0 \; \text{as} \; t \to \infty. \]
Global version of Lasalle's Theorem

Consider the system
\[ \dot{x} = f(x), \ x_0 = 0 \]
Let \( v(x) \) be a PDF with \( \dot{v} \leq 0 \) \( \forall x \in \mathbb{R}^n \)
If the set
\[ S = \{ x \in \mathbb{R}^n / \dot{v}(x) = 0 \} \]
contains no invariant sets other than the origin, then the origin is globally asymptotically stable.

Lasalle's Theorem for Periodic Systems

Consider the system
\[ \dot{x} = f(x, t), \ x_0 = 0 \]
where \( f \) is periodic
\[ f(x, t) = f(x, t + T) \] \( \forall t \in \mathbb{R}^n \)
Further, let \( v(x, t) \) be a PDF which is periodic in \( t \) with period \( T \). Define:
\[ S = \{ x \in \mathbb{R}^n : \dot{v}(x, t) = 0 \} \text{ and } T \geq 0 \]
Then if \( \dot{v}(x, t) \leq 0 \) \( \forall x \in \mathbb{R}^n \) \( \forall t \geq 0 \) and \( S \) contains no invariant sets other than the origin, then the origin is globally a.s.
Generalization of Lasalle's Theorem:

A difficulty arises in extending Lasalle's Theorem to arbitrary time-varying systems, which is that

\[ \{ x : \dot{V}(x,t) = 0 \} \]

may be a time-varying set.

However, if we can assume that:

\[ \dot{V}(x,t) \leq -W(x) \leq 0 \]

Then the set \( S \) may be defined as:

\[ \{ x : W(x) = 0 \} \]

and Lasalle's Theorem may be generalized as follows:

Consider \( f(x,t), x \in S \), and suppose that for \( x \in B_r \) (a ball of radius \( r \) around \( x \)), there exists a function \( V(x,t) \) such that for functions \( x_1, x_2 \) of class \(-K\):

\[ x_1(1/x_1) \leq V(x,t) \leq x_2(1/x_1) \]

Also, assume that for some
non-negative function \( w(x) \)

\[
\dot{v}(x,t) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f(x,t) \leq -w(x) \leq 0
\]

Then, for all \( \|x(t_0)\| \leq x^{-1}_2(\alpha, r) \)
the trajectories \( x(\cdot) \) are bounded and

\[
\lim_{t \to \infty} w(x(t)) = 0
\]

\[\text{[meaning that } x(t) \text{ approaches a} \]
set \( E \) defined by:

\[
E := \{x \in \mathbb{B}_r : w(x) = 0\}.
\]
Lyapunov Theory. **Example.**

Nonlinear Spring:

\[ M \ddot{x} = -F_s \quad \text{where} \quad F_s = k(x) \]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-\frac{1}{M} K(x_1)
\end{bmatrix}
\]

Example: Duffing equation, no friction

Nonlinear spring \( F_s = k(x) \) restoring: \( k(x) x \geq 0 \) for all \( x \)
\[ v(x) = \frac{1}{2} m x_2^2 + \int_0^{x_1} k(y) \, dy \]

\[ \text{total energy} \]

\[ K.E + P.E. \]

\[ \Rightarrow v(x) = 0 \]

\[ \Rightarrow \frac{\partial v(x)}{\partial x} = 0 \]

\[ \therefore v(x) \leq 0. \]

So if \( v(x) \) is LDP or PD we can determine the stability of the system.

\[ v(x) = \frac{1}{2} m x_2^2 + \int_0^{x_1} k(y) \, dy \]

\[ \geq 0 \]

\[ \geq 0 \text{ since restoring spring (i.e. } x, k(x) \geq 0) \]

\[ \therefore v(x) \geq 0. \]

\[ \text{(semi-PD)} \]

When is \( v(x) \) LDP or PD? When \( x_1 k(x_1) \geq 0 \) and when the spring has no dead zone at 0. ie. cannot have the situation:

\[ k(x) = 0 \text{ when } x \neq 0 \]