GOALS OF THIS LECTURE:

- Lyapunov Instability Theorem
- Lyapunov Theory for Linear Systems
- The Indirect Method of Lyapunov (Stability from Linearization)
  ... beyond Hartman-Grobman
- Lyapunov Exponential Stability Theorem

REFS:

SASTRY  95.3.4, 5.6, 5.7, 5.8, 5.9
KHALIL  94.6, 4.7, 4.3
Lyapunov Instability Theorem (§ 5.6 Sastry)

\( xe = 0 \) is an unstable equilibrium point of \( \dot{x} = f(x, t) \) at time \( t_0 \) if there exists a decreasing function \( V: \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R} \) such that

1. \( \dot{V}(x, t) \) is a LPDF
2. \( V(0, t) = 0 \) and there exist points \( x \) arbitrarily close to 0 such that \( V(x, t_0) > 0 \).

Remarks

- insists on \( \dot{V}(x, t) \) being an LPDF so as to have a mechanism for the increase of \( V \)
- however, since we do not need to guarantee that every initial condition close to the origin is repelled from the origin, we do not need to assume that \( \dot{V} \) is a LPDF.
Sketch of proof (Lyapunov Instability)

$v(x,t)$ is decreasing $\Rightarrow v(x,t) \leq \psi(\|x\|)$ \(\quad 1\)
$v(x,t)$ is LPDF $\Rightarrow \dot{v}(x,t) \geq \phi(\|x\|)$ \(\quad 2\)

Suppose \(1\) holds \(\forall x \in B_r(x_e)\)
"  
Suppose \(2\) holds \(\forall x \in B_s(x_e)\)

We need to show that for some $\delta, \delta > 0$, there is no $S$ such that

$\|x_0 - x_e\| < S \Rightarrow \|x(t) - x_e\| < \delta \quad \forall t \geq t_0.$

So choose $\delta = \min(s,r)$.

Given any $\delta > 0$, choose $x_0$ such that

$\|x_0 - x_e\| < \delta$ and $v(x_0,t_0) > 0$.

Thus, as long as $x(t) \in B_{\delta}$ then $v(x,t) > 0$.

$\Rightarrow v(x,t) > v(x_0,t_0) > 0$

$\Rightarrow \|x(t) - x_e\|$ is bounded away from zero
(since $v(x_e,t) = 0$)

$\Rightarrow \dot{v}(x(t),t)$ is bounded away from zero

$\Rightarrow v(x,t)$ will exceed $B_{\delta}$ in finite time (since $\dot{v} > 0$)

$\Rightarrow \|x(t) - x_e\| > \delta$ in finite time.
example

\[ \dot{x}_1 = -x_2 + x_1 (x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \]
\[ \dot{x}_2 = x_1 + x_2 (x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \]

Prove that \((0,0)\) is unstable.

Choose \( \mathcal{V}(x) = \|x\|^2 \) . PDF
\[ \dot{\mathcal{V}}(x) = 2 \|x\| \|x\|^4 \sin(\|x\|^2) \] . LPDF

\[ \therefore \text{by the Lyapunov Instability Theorem} \]

The origin is unstable.

Remark: Note that in this case \( \mathcal{V}(x) \) is a PDF, and thus condition 2. of the Lyapunov Instability Theorem is automatically satisfied!
Lyapunov Theory for Linear Systems

\[ \dot{x} = Ax \quad x \in \mathbb{R}^n \quad (L) \]

\( x(0) = 0. \)

- helps in understanding the construction of Lyapunov functions for nonlinear systems
- preparation for Lyapunov's Indirect Method

**Stability of LTI Systems**

1. The equilibrium at the origin of (L) is stable iff all the eigenvalues of \( A \) are in the closed left half plane (written \( \mathbb{C}_- \)) and those on the \( \text{jw-axis} \) are distinct.

2. The equilibrium at the origin of (L) is asymptotically stable iff all the eigenvalues of \( A \) are in the open left half plane (written \( \mathbb{C}_- \)).
Alternatively, we can construct a Lyapunov function

\[ V(x) = x^T P x \]

where \( P > 0 \), \( P \in \mathbb{R}^{n \times n} \), symmetric

\[ \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x \]

\[ = x^T (AP + PA) x \]

Therefore, \(-\dot{V}(x)\) is P.D.F. if there exists a \( Q > 0 \), symmetric, such that

\[ A^T P + PA = -Q \]

"Lyapunov Equation"

\[ \text{THEOREM} \]
Lyapunov Stability for Linear Time-Invariant Systems.

\[ \dot{x} = Ax \]

is asymptotically stable iff for any given \( Q \in \mathbb{R}^{n \times n}, Q > 0, \) symmetric, there exists a \( P \in \mathbb{R}^{n \times n}, P > 0, \) symmetric, that satisfies the Lyapunov equation

\[ A^T P + PA = -Q \]

\[ \text{PROOF} \]
If, for given \( Q > 0 \), symmetric, there exists a \( P > 0 \), symmetric
Such that \( A^T P + PA = -Q \), then using the Lyapunov function

\[ V(x) = x^T P x \quad (P.D.F.) \]

where \( -\dot{V}(x) = x^T Q x \quad (P.D.F.) \)

We prove that \( \dot{x} = Ax \) is asymptotically stable.

Conversely, if \( A \) has all of its eigenvalues in \( C^\circ \), then given \( Q > 0 \), symmetric, define

\[ P = \int_0^\infty e^{A^T r} Q e^{A r} \, dr \]

Claim \( P \) is symmetric and positive definite (you should convince yourself of this).

Claim \( P \) solves the Lyapunov equation

To show this:

\[ A^T P + PA = \int_0^\infty e^{A^T r} Q e^{A r} \, dr \]

\[ + \int_0^\infty e^{A^T r} Q e^{A r} A \, dr \]
\[
\begin{align*}
&= \int_0^\infty \frac{d}{dt} (e^{A^T r} Q e^{A r}) \, dt \\
&= e^{A^T r} Q e^{A r} \bigg|_0^\infty = -Q \tag{\text{Remark}}
\end{align*}
\]

Remark: The extension of Lyapunov stability theory to linear time-varying systems \( \dot{x} = A(t)x \), \( x \in \mathbb{R}^n \), \( x(t_0) = x_0 \) requires the definition of a state transition matrix \( \Phi(t, t_0) \):

\[
\Phi(t, t_0) = A(t) \Phi(t, t_0), \quad \Phi(t_0, t_0) = I
\]

Solutions to \( \dot{x} = A(t)x \), \( x(t_0) = x_0 \) are then written as

\[
x(t) = \Phi(t, t_0) x(t_0)
\]

and the Lyapunov stability theorem (LTV):

If \( A(\cdot) \) is bounded, and for some \( Q(t) \geq \alpha I \), \( P(t) := \int_0^\infty \Phi^T(r, t) Q(r) \Phi(r, t) \, dr \) is bounded, then \( x^* = 0 \) is an asymptotically stable equilibrium point.
**Example** Consider a second-order linear system whose A matrix is

\[
A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}
\]

Take \( Q = I \) and denote \( P \) by

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]

where, due to symmetry of \( P \), \( P_{21} = P_{12} \).

Then, the Lyapunov equation is:

\[
\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = I
\]

whose solution is:

\[
P_{11} = \frac{5}{16}
\]

\[
P_{12} = P_{22} = \frac{1}{16}
\]

\[
\therefore P = \frac{1}{16} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}
\]

is positive definite, and therefore the linear system is globally asymptotically stable.
Theorem: The Indirect Method of Lyapunov (Stability from Linearization)

Let $x = 0$ be an equilibrium point of the nonlinear system

$$\dot{x} = f(x)$$

where $f : D \rightarrow \mathbb{R}^n$ is continuously differentiable and $D$ is a neighborhood of the origin.

Let

$$A = \frac{2f(x)}{dx} \bigg|_{x=0}$$

Then:

1. The origin is locally asymptotically stable if $\Re \lambda_i < 0$ for all eigenvalues of $A$.

2. The origin is unstable if $\Re \lambda_i > 0$ for one or more of the eigenvalues of $A$.

Remark: Not as fine a classification of equilibrium stability type as Hartman-Grobman.

... HOWEVER ...
Remark 2: We can couple this theorem with the Lyapunov Stability Theorem for LTI systems to estimate the domain of attraction of an equilibrium point: D.O.A.

Consider $\dot{x} = f(x)$ with equilibrium $x^*$. The domain of attraction of $x^*$ is the set of all initial conditions $x_0$ of $\dot{x} = f(x)$ which converge to $x^*$.

How do we estimate this domain of attraction? Let's look at a partial proof of Lyapunov's Indirect Theorem:

Partial proof:

Assume $\rho_d < 0$ for all eigenvalues of $A$. Then, by the Lyapunov Stability Theorem for LTI systems, we know that for $Q > 0$, $Q$ symmetric, the solution $P$ of

$$A^TP + PA = -Q$$

is such that $P > 0$. 

Consider \( V(x) = x^T P x \) as a Lyapunov function candidate for the nonlinear system \( \dot{x} = f(x) \):

\[
\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x \\
= x^T P f(x) + f(x)^T P x
\]

now let \( f(x) = Ax + g(x) \)
where \( A = \frac{\partial f}{\partial x} \bigg|_{x=0} \)

and \( g(x) \) are higher order terms, so that \( \frac{\|g(x)\|}{\|x\|} \to 0 \) as \( \|x\| \to 0 \)

\[
\therefore \dot{V}(x) = x^T P (Ax + g(x)) + (x^T A^T + g^T(x))Px \\
= x^T (PA + A^TP)x + 2x^TPg(x) \\
= -x^T Q_x x + 2x^TPg(x)
\]
Therefore, for any $\gamma > 0$ there exists an $r > 0$ such that
\[ \|g(x)\| < \gamma \|x\| \text{ for all } \|x\| < r. \]

So going back to our equation for $\dot{V}(x)$:
\[
\dot{V}(x) = -x^T Q x + 2x^T P g(x) \\
\leq -x^T Q x + 2\gamma \|P\| \|x\|^2
\]
for $\|x\| < r$

but $x^T Q x \geq \lambda_{\min}(Q) \|x\|^2$

minimum eigenvalue of $Q$.

\[
\text{[since } x^T Q x = x^T T \Lambda T^T x \\
= (T x)^T \Lambda (T x) \\
\geq \lambda_{\min}(Q) \|T x\|^2 \\
= \lambda_{\min}(Q) \|x\|^2 \text{ ]}
\]

So finally...
\[
\dot{V}(x) \leq -[\lambda_{\min}(Q) - 2\gamma \|P\|] \|x\|^2
\]
for $\|x\| < r$.

\[ \therefore \text{choose } \gamma \text{ to ensure } -\dot{V}(x) \text{ LPDF on } \|x\| < r. \]

\[ B_r = \{ x : \|x\| < r \} \text{ is a conservative estimate of D.O.A.} \]
Lyapunov Exponential Stability Theorem: its Converse

Previous Lyapunov Theorems don't give us any information about the rate of convergence of solutions to the equilibria... with more information we can determine this.

Theorem (Lyapunov Exponential Stability Theorem)

\( x_e \) is a locally exponentially stable equilibrium point of
\[
\dot{x} = f(x, t)
\]
if and only if

There exists a function \( v(x, t) \) and some constants \( h, \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \) such that for all \( x \in \mathbb{R}^n, \ t \geq 0 \)
\[
\begin{align*}
\alpha_1 \| x \|^2 & \leq v(x, t) \leq \alpha_2 \| x \|^2 \\
\dot{v}(x(t), t) & \leq -\alpha_3 \| x \|^2 \\
\left| \frac{\partial v(x, t)}{\partial x} \right| & \leq \alpha_4 \| x \| 
\end{align*}
\]