Goals of this lecture:

- The indirect method of Lyapunov for linear time-varying systems.
- Proof of the Lyapunov Exponential Stability Theorem.

Refs  SAstry  §5.8, 5.9, 5.3.4.
We now consider the Indirect Method of Lyapunov for time-varying systems, extending our work from Lecture Notes 15, page 15-9:

Consider $\dot{x} = f(x, t)$ with $f(0, t) = 0 \forall t \geq 0$

Define $A(t) = \left. \frac{\partial f(x, t)}{\partial x} \right|_{x=0}$

and $f_1(x, t) = f(x, t) - A(t)x$

We will need to assume that the "remainder" $f_1(x, t)$ is uniformly of order $x^2$, i.e. that

$$\lim_{|x| \to 0} \sup_{t \geq 0} \frac{|f_1(x, t)|}{|x|} = 0$$

*Exercise: Show that for $f(x, t) = -x + tx^2$, the remainder is not uniformly of order $x^2$.

With this assumption, we can define the linearization of $f(x, t)$ about $x = 0$ as

$$\dot{z} = A(t)z.$$
Theorem (indirect method of lyapunov for linear time-varying systems).

Let \( \dot{x} = f(x,t) \) and let us assume that
\[
\lim_{|x| \to 0} \sup_{t \geq 0} \frac{|f_1(x,t)|}{|x|} = 0.
\]

Also, assume that \( A(t) \) is bounded. If \( 0 \) is a uniformly asymptotically stable equilibrium point of
\[
\dot{x} = A(t)x.
\]

Then it is a locally uniformly asymptotically stable equilibrium point of \( \dot{x} = f(x,t) \).

Proof: Since \( A(\cdot) \) is bounded and \( 0 \) is a uniformly asymptotically stable equilibrium point of \( \dot{x} = A(t)x \),

then from our study of lyapunov functions for linear time-varying systems (16-8, 16-9):
\[
P(t) = \int_0^\infty \phi(r) \phi^T(r,t) \, dr
\]
satisfies
\[
\beta x^T x \geq x^T P(t) x \geq \alpha x^T x
\]
for some \( \alpha, \beta > 0 \).

Thus, \( v(x,t) = x^T P(t) x \) is a decrescent PDF.

Also,
\[
\dot{v}(x,t) = x^T [\dot{P}(t) + A(t)P(t) + P(t)A(t)] x
+ 2x^T P(t) f_1(x,t)
= -x^T x + 2x^T P(t) f_1(x,t).
\]
Now, since \( \lim_{|x| \to 0} \sup_{t \geq 0} \frac{|f_i(x,t)|}{|x|} = 0 \), for \( r > 0 \) such that \( |f_i(x,t)| \leq \frac{1}{3} |x| \) \( \forall x \in B_r \) \( \forall t \geq 0 \),

now we know that \( x^TP(t)x \leq \beta x^Tx \).

Thus \( |2x^TP(t)f_i(x,t)| \leq \frac{2|x|^2}{3} \) \( \forall x \in B_r \),

\[ \therefore \quad \dot{V}(x,t) \leq -\frac{1}{3} |x|^2 \quad \forall x \in B_r. \]

\[ \therefore \quad -\dot{V}(x,t) \text{ is a PDF locally.} \]

\[ \Rightarrow \quad 0 \text{ is uniformly asymptotically stable in } B_r. \]

**Remarks**

1. The theorem requires uniform asymptotic stability of the linearized system.

2. If the linearization is time-invariant, then \( \dot{A}(t) = A \) and if \( \sigma(A) \subset \mathbb{C}^c \) when the non-linear system is uniformly asymptotically stable.

3. We can use the estimates of this theorem to give a conservative bound on the domain of attraction (D.O.A.) of the origin:

\[ \dot{V}(x,t) \leq -\frac{x^Tx}{3} \quad \forall x \in B_r. \]
Since \( \beta \|x\|^2 \geq \nu(x, t) \geq \alpha \|x\|^2 \), we have that
\[
\nu(x, t) \leq -\frac{1}{3\alpha} \nu(x, t)
\]
\[
\Rightarrow \nu(x, t) \leq \nu(x(t_0), t_0) e^{-\frac{1}{3\alpha}(t-t_0)}
\]
\[
\Rightarrow \alpha \|x(t)\|^2 \leq \beta \|x(t_0)\|^2 e^{-\frac{1}{3\alpha}(t-t_0)}
\]
\[
\Rightarrow \|x(t)\| \leq \sqrt{\frac{\beta}{\alpha}} e^{-\frac{1}{6\alpha}(t-t_0)} \|x(t_0)\|
\]
i.e. if \( \|x(t_0)\| < \sqrt{\frac{\alpha}{\beta}} r \) then \( \|x(t)\| < r \)

Finally, we prove the Lyapunov Exponential Stability Theorem and its Converse:

(proof for Theorem on p. 15-13):
\( x = 0 \) is a locally exponentially stable equilibrium point of
\[
\dot{x} = f(x, t)
\]
iff
\[
\exists \nu(x, t), \exists \tau, \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \text{ such that } \forall x \in B_r, \ t > t_0
\]
\[
\alpha_1 \|x\|^2 \leq \nu(x, t) \leq \alpha_2 \|x\|^2
\]
\[
\dot{\nu}(x(t), t) \leq -\alpha_3 \|x\|^2
\]
\[
\left| \frac{\partial \nu}{\partial x} (x, t) \right| \leq \alpha_4 \|x\|
\]
Proof (of IF direction; other direction given in text):

label inequalities \( 0, 2, 2 \) as shown.

from \( 3, 2 \): \( \dot{V}(x(t), t) \leq -\alpha_3 |x|^2 \leq -\frac{\alpha_3}{\alpha_2} V(x, t) \)

\[ \Rightarrow V(x, t) \leq V(x_0, t_0) e^{-\alpha_3/\alpha_2 (t-t_0)} \]

now use \( 4 \):

\[ \Rightarrow \alpha_1 |x|^2 \leq \alpha_2 |x_0|^2 e^{-\alpha_3/\alpha_2 (t-t_0)} \]

\[ \Rightarrow |x| \leq m |x_0| e^{-\alpha (t-t_0)} \]

where \( m = \sqrt{\frac{\alpha_2}{\alpha_1}} \), \( \alpha = \frac{\alpha_3}{2\alpha_2} \)

\[ \Rightarrow \text{locally exponential stability of } x(t) = 0. \]