Goals of this lecture:

- Introduce nonlinear control:
  - Linearization by state feedback.
- Relative degree
- Overall control topology.

Refs:

SASTRY  G.9.1, G.2
KHALIL  Ch. 13.
Linearization by State Feedback:


Main Idea: Given a nonlinear system of the form
\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \quad \text{not as general} \\
y &= h(x)
\end{align*}
\]
use (nonlinear) state feedback \( u = k(x) + v \) to make the system EXACTLY linear

- not the same as Jacobian linearization approx. linear.
EXAMPLE 1. Linear, Time-Invariant
Single-Input Single-Output (SISO)
\( y \in \mathbb{R} \)
\( u \in \mathbb{R} \)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\alpha_4 x_4 - \alpha_3 x_3 - \alpha_2 x_2 - \alpha_1 x_1 + bu \\
y &= c x_1, \\
\end{align*}
\]
\( \alpha_1, c \in \mathbb{R} \).

a) Find a memoryless state feedback of the form
\( u = kx + u_{(\text{new input})} \),
so that all of the eigenvalues are at \( s = -1 \).

SOLUTION:

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
0 \\
b
\end{bmatrix} u
\]

\[
A, B
\]

\( \Rightarrow \) system is controllable for \( b \neq 0 \)

(Check rank \([B|AB|A^2B|A^3B]\))
For all eigenvalues at \( s = -1 \) in the closed loop system, we must have

\[
\bar{A} = A + \begin{bmatrix} 0 & 0 & 0 & k_4 \\ 0 & 0 & k_3 & k_4 \\ 0 & k_2 & k_3 & k_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}
\]

where \( \bar{A} \) has characteristic polynomial:

\[
(s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1
\]

\[
\therefore \bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix}
\]

\[
\begin{align*}
\beta_1 &= -\alpha_1 + bk_1 \\
\beta_2 &= -\alpha_2 + bk_2 \\
\beta_3 &= -\alpha_3 + bk_3 \\
\beta_4 &= -\alpha_4 + bk_4
\end{align*}
\]

b) Now suppose that the equation for \( \dot{x}_4 \) in the original system is replaced by \( \dot{x}_4 = \Psi(x_1, x_2, x_3, x_4) + bu \) where \( \Psi(\cdot, \cdot, \cdot, \cdot) \) is known, nonlinear. Use a memoryless state feedback of the form

\[
\begin{align*}
\mathbf{u} &= K(x) + \nu \\
&\quad \text{nonlinear}
\end{align*}
\]
... so as to make the input-output transfer function from \( v \) to \( y \) 

\[
\frac{cy}{(s+1)^4}.
\]

**Solution:**

\[
\dot{x}_4 = y(x_1, x_2, x_3, x_4) + bu
\]

Choose \( u = -\frac{y}{b} (x_1, x_2, x_3, x_4) + Kx + v \)

\[
\therefore \dot{x}_4 = bKx + bv \quad \text{linear in } x, v \]

\[
\therefore \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ bk_1 & bk_2 & bk_3 & bk_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} v
\]

\[
\therefore \quad \text{here } \quad k_1 = \frac{-1}{b}, \quad k_2 = \frac{-4}{b}, \quad k_3 = \frac{-6}{b}, \quad k_4 = \frac{-4}{b}.
\]

Check: the transfer function from \( v \) to \( y \):

\[
y = cx_1,
\]

now \( \dot{x}(t) = Ax(t) + Bu(t) \) where \( B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} \)

\[
\therefore (sI - A)x(s) = Bu(s)
\]

\[
\therefore x(s) = (sI - A)^{-1}Bu(s)
\]

and \( y(s) = cx_1(s) = [c \ 0 \ 0 \ 0]x(s) \)
\[
\frac{y(s)}{v(s)} := G(s) = [c \ 0 \ 0 \ 0] (sI - \bar{A})^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} = \frac{cb}{(s+1)^4}
\]

\[\]

**Example 2:** Consider the following single-input, single-output nonlinear system

\[
\begin{align*}
\dot{x}_1 &= \xi_2 \\
\dot{x}_2 &= \xi_3 \\
\dot{x}_3 &= \xi_4 \\
\vdots \\
\dot{x}_r &= a(\xi, \eta) + b(\xi, \eta) u \\
\eta &= q(\xi, \eta).
\end{align*}
\]

**Output:** \( y = \xi_1 
\]

\( u \in \mathbb{R}, y \in \mathbb{R}, \quad \xi := \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \end{bmatrix} \in \mathbb{R}^r, \quad \eta \in \mathbb{R}^{n-r} \)

and \( a(\xi, \eta), \ b(\xi, \eta), \ q(\xi, \eta) \) are all smooth, scalar functions of \( \xi \) and \( \eta \):

ie. \( a(\cdot, \cdot): \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R} \) and also, \( b(\xi, \eta) \neq 0 \).
Consider the state feedback law

\[ u = -\frac{a(\xi, \eta) + v}{b(\xi, \eta)} \]

With this state feedback law, find the relationship between \( y \) and \( u \). What is remarkable about it? Is the closed loop system observable?

**Solution:**

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= a(\xi, \eta) + b(\xi, \eta) \left[ -\frac{a(\xi, \eta) + v}{b(\xi, \eta)} \right] \\
\dot{\eta} &= q(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

We note that:

1. \( \eta \) does not affect \( \xi \).
2. We can compute \( \hat{y}(s) = \frac{1}{s^r} \).

ie. The feedback law above was chosen so that the I/O relationship is described by a linear system!

Now, suppose \( u(t) = 0 \), \( \xi_1(0) = \xi_2(0) = \ldots = \xi_r(0) = 0 \) and \( \eta(0) \neq 0 \).
from the above, we obtain:
\[ y(t) = 0 \quad \forall t > 0 \]

Therefore, the states \( n \) are not observable \( \Box \)

**Input-output linearization for SISO Systems.**

Given
\[
\begin{align*}
x &= f(x) + g(x) \, u \\
y &= h(x)
\end{align*}
\]

with \( x \in \mathbb{R}^n \), \( \mathbb{R} \), \( y \in \mathbb{R} \)

\( f(x) \) \( \text{smooth} \) (infinitely differentiable or \( C^\infty \))

\( g(x) \) \( \text{smooth function from} \ \mathbb{R}^n \ \text{to} \ \mathbb{R} \).

Let \( x^* \) be an equilibrium point of the undriven \( (u=0) \) system, i.e. \( f(x^*) = 0 \).

Let the following calculations be for \( x \in U = B_r(x^*) \).

Differentiating \( y \) with respect to time:

\[
y = \frac{\partial h}{\partial x} \, \dot{x}
\]

\[
= \frac{\partial h}{\partial x} \left[ f(x) + g(x) \, u \right]
\]

\[
= \frac{\partial h}{\partial x} \cdot f(x) + \frac{\partial h}{\partial x} \cdot g(x) \, u \quad (\ast)
\]
We've seen this type of derivative before (Lyapunov Theory):

\[ \frac{\partial h}{\partial x} \cdot f(x) =: L_f h(x) \]

"Lie derivative of \( h \) with respect to \( f \)"

\[ \frac{\partial h}{\partial x} \cdot g(x) =: L_g h(x) \]

"Lie derivative of \( h \) with respect to \( g \)"

- if \( |L_g h(x)| > \delta \), \( \delta > 0 \) "bounded away from zero" for all \( x \in U \), then the state feedback law is given by

\[ u = \frac{1}{L_g h(x)} \left[ -L_f h(x) + v \right] \]

- if \( L_g h(x) \equiv 0 \), meaning \( L_g h(x) = 0 \) \( \forall x \in U \), we differentiate (\(*\)) again:

\[ \ddot{y} = \frac{\partial L_f h}{\partial x} f(x) + \frac{\partial L_f h}{\partial x} g(x) \]

\[ =: L_f^2 h(x) + L_g L_f h(x) u \]
if $|\log f h(x)| > S_2, S_2 > 0 \; \forall x \in U$ then the state feedback law is given by

$$u = \frac{1}{\log f h(x)} (-f^2 h(x) + v)$$

ALGORITHM:

given \quad \dot{x} = f(x) + g(x) u \quad y = h(x)

1) differentiate $y$ with respect to time.

$$\dot{y} = f h(x) + g h(x) u$$

2) if $|\log h(x)| > S_1$ for all $x \in U$, let

$$u = \frac{1}{\log h(x)} [-f h(x) + v]$$

I/O linear system is $\ddot{y} = v$

3) if $\log h(x) = 0$, differentiate again

$$\ddot{y} = f^2 h(x) + g f h(x) u$$

4) if $|\log f h(x)| > S_2$ for all $x \in U$, let

$$u = \frac{1}{\log f h(x)} [-f^2 h(x) + v] \Rightarrow \ddot{y} = v$$

5) if $\log f h(x) = 0$, differentiate again.
More generally, if $\gamma$ is the smallest integer for which $\lg f^i h(x) = 0$ on $U$ for $i = 0, \ldots, \gamma - 2$ and $\lg f^{\gamma - 1} h(x)$ is bounded away from zero on $U$, then the control law given by

$$U = \frac{1}{\lg f^{\gamma - 1} h(x)} (-f^\gamma h(x) + v)$$

yields the $\gamma$th-order linear system from input $v$ to output $y$:

$$y^\gamma = v.$$

**Defn** Strict Relative Degree

The SISO nonlinear system

$$\dot{x} = f(x) + g(x) u$$
$$y = h(x)$$

is said to have strict relative degree $\gamma$ at $x^* \in U$ if

$$\lg f^i h(x) = 0 \quad \forall x \in U, \quad i = 0, \ldots, \gamma - 2$$
$$\lg f^{\gamma - 1} h(x^*) \neq 0$$
Overall block diagram:

\[ u = k(x, u) \]
\[ \dot{x} = f(x) + g(x)u \]
\[ y = h(x) \]

linearization loop
Consider the two-input, two-output case:

\[
\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2
\]

\[y_1 = h_1(x) \quad x \in \mathbb{R}^n\]

\[y_2 = h_2(x) \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2\]

\[y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2\]

Start with \(y_1\): differentiate \(y_1\) until an input appears, let \(\gamma_1\) be the smallest integer such that at least one of the inputs appears in \(y_1^{(\gamma_1)}\). \(y_1^{(\gamma_1)}\) is the \(\gamma_1\)th derivative of \(y_1\).

Repeat for \(y_2\), with \(\gamma_2\) the smallest integer such that at least one of the inputs appears in \(y_2^{(\gamma_2)}\).

Thus, \(y_1^{(\gamma_1)} = Lf^{\gamma_1}h_1 + Lg_1Lf^{\gamma_1-1}h_1u_1 + Lg_2Lf^{\gamma_1-1}h_1u_2\)

\(y_2^{(\gamma_2)} = Lf^{\gamma_2}h_2 + Lg_1Lf^{\gamma_2-1}h_2u_1 + Lg_2Lf^{\gamma_2-1}h_2u_2\)
Let \( A(x) = \begin{bmatrix} l g_1 l f^{x_1-1} h_1 & l g_2 l f^{x_1-1} h_1 \\ l g_1 l f^{x_2-1} h_2 & l g_2 l f^{x_2-1} h_2 \end{bmatrix} \)

**Definition**  Vector Relative Degree

The system \((NL-TITO)\) is said to have vector relative degree \( x_1, x_2 \) at \( x^* \) if \( l g_i l f^k h_i(x) = 0 \), \( 0 \leq k \leq x_i - 2 \) for \( i = 1, 2 \) and the matrix \( A(x_0) \) is non-singular.

Then, the state feedback law

\[
 u = - A^{-1}(x) \begin{bmatrix} l f^{x_1} h_1 \\ l f^{x_2} h_2 \end{bmatrix} + A^{-1}(x) u
\]

yields a linear closed loop system

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]