GOALS OF THIS LECTURE:

- Poincaré–Bendixson Thm.
  mainly used as a quantitative tool to determine "where trajectories end up"

- Lotka–Volterra example

REFS: SASTRY §2.3, 2.2
      Khalil §2.6

Defn [Flow] The state of the system
\[ \dot{x} = f(x) \] (NL)
at time \( t \) starting from \( x \) at time 0
is called the flow and is denoted by \( \Phi_t(x) \).

Defn [W-limit set] A point \( z \in \mathbb{R}^2 \) is said to
be an \( W \)-limit point of a trajectory \( \Phi_t(x) \) of
(NL) if there exists a sequence of times
\( t_n, n = 1 \ldots \infty \) such that \( t_n \to \infty \) as \( n \to \infty \)
for which \( \lim_{n \to \infty} \Phi_{t_n}(x) = z \). The set of all
limit points of \( \Phi_t(x) \) is called the \( W \)-limit set,
Poincaré-Bendixson Theorem

Consider the planar dynamical system:
\[ \begin{align*}
\dot{x}_1 &= f_1 (x_1, x_2) \\
\dot{x}_2 &= f_2 (x_1, x_2)
\end{align*} \text{ (NL)}

Let \( M \) be a compact, positively invariant set for the flow \( \Phi_t (x) \). Let \( p \in M \). Then, if \( M \) contains no equilibrium points, \( \omega(p) \) is a closed orbit of (NL).

(More simply:) Every compact (non-empty) positively invariant set \( M \) contains an equilibrium point or a closed orbit.

Remark: if \( M \) contains equilibrium points, \( M \) may also contain a union of trajectories connecting these equilibria

\[ \text{e.g.,} \]

Recall that this is not a closed orbit.
Remark 2: if $\gamma$ is a closed orbit of NL, enclosing an open set $U$, then $U$ contains an equilibrium point (check using index theory), and $U$ may also contain a closed orbit.
**Worked Example:** Lotka-Volterra predator-prey model

\[
\begin{align*}
\dot{x} &= (a - by - \lambda x)x \\
\dot{y} &= (cx - d - \mu y)y
\end{align*}
\]

\(a, b, c, d, \lambda, \mu \geq 0\)

**Equilibria:**

\( (0, 0) \)

\( \left( \frac{a}{\lambda}, 0 \right) \)

\( (0, \frac{-b}{\mu}) \rightarrow \text{disallowed since } x \geq 0, \ y \geq 0 \)

\( e_3 = \left( \frac{bd + a\mu}{bc + \lambda\mu}, \frac{ac - d\lambda}{bc + \lambda\mu} \right) \rightarrow \text{disallowed for } ac < d\lambda \)

**Jacobian:**

\[
J = \begin{bmatrix}
    a - by - 2\lambda x & -bx \\
e\gamma & cx - d - 2\mu y
\end{bmatrix}
\]

at \((0, 0)\), \( J = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix} \rightarrow \text{saddle} \)

at \( \left( \frac{a}{\lambda}, 0 \right) \), \( J = \begin{bmatrix} -a & -\frac{ba}{\lambda} \\ 0 & \frac{ca - d}{\lambda} \end{bmatrix} \rightarrow \text{e-vals} \)

at \( e_3 \), big mess but do-able.
CASE 1: \( ac < dA \Rightarrow 2 \) equilibria
\((0,0) \quad (\frac{a}{A},0)\)

(see Figure)

Remarks: \(1) \) it is impossible for both prey \& predator populations to increase at once:
\ie we cannot have \( x > 0, y > 0 \)

2) The region \( \text{I} \cup \text{II} \) is invariant (verify). Also, it cannot contain any limit cycles (more generally, closed orbits) since \( y \leq 0 \) throughout \( \text{I} \cup \text{II} \).

Therefore, by Poincaré–Bendixson all trajectories starting in \( \text{I} \cup \text{II} \) will tend to \((0,0) \) or \((\frac{a}{A},0)\), the only equilibria.

3) Trajectories starting in \( \text{III} \) eventually enter \( \text{II} \) since \( x < 0, y > 0 \) in \( \text{III} \)

\[ \vdots \]

\( 2) \; 3) \) Show that all non-zero initial predator \& prey populations tend to \((\frac{a}{A},0)\).
CASE 1: \[ a = 2, \ b = 0.5, \ c = 1, \ d = 2, \]
\[ \lambda = 2, \ \mu = 2 \]
CASE 2: \( ac > dd \Rightarrow 3 \) equilibria \((0,0), \left(\frac{a}{d}, 0\right), z\)

where \( z = \left( \frac{bd + am}{bc + m}, \frac{ac - dd}{bc + m} \right) \)

(see Figure on next page)

\( z \) is a stable \{focus \}

node

\((0,0)\) is still a saddle.

\( \left(\frac{a}{d}, 0\right) \) is a saddle

The region \( D \) (rectangular region in the Figure) is invariant.

Thus it could contain closed orbits (by Poincaré-Bendixson) and they must surround \( z \) (by Index Theory).

Remarks

1) All initial conditions starting outside \( D \) eventually enter \( D \)

2) Trajectories inside \( D \) will either tend to a \underline{limit cycle} or \( z \)

(\( \text{or} \left(\frac{a}{d}, 0\right) \text{ or } (0,0) \text{ if they start at these points} \)
CASE 2

\[ a = 2, \quad b = 0.5, \quad c = 1, \quad d = 2 \]
\[ \lambda = 0.5, \quad \mu = 0.5 \]
Conclusions:

- Invariant sets are often easy to find in \( \mathbb{R}^2 \)
- Combining our tools:
  - equilibria/linearization
  - Bendixson
  - Poincaré-Bendixson \{ only work in \( \mathbb{R}^2 \) \}
  - Index Theory

We can quickly determine the behavior of the system without doing any simulations!