GOALS OF THIS LECTURE:

- define Describing Functions
- compute describing functions for single-valued and double-valued skew symmetric functions (i.e. relays, relays with hysteresis)

REFS: SASTRY §4.1
Describing functions are used in NL analysis. Sometimes, too complicated to fully analyze.

Describing function analysis is an approximation technique, often useful in engineering practice, which may be used to predict the frequency and amplitude of oscillations in systems involving nonlinearities.

IE:

![Diagram](image)

IDEA: Determine the "Transfer Function" between the input to $f$ and the first harmonic (of the input) in the output of $f$. 
\[ f(\text{as} \sin \omega t) = \sum_{k=0}^{\infty} (a_k(n) \sin k \omega t + b_k(n) \cos k \omega t) \]

\[ \Rightarrow b_0(n) + a_1(n) \sin \omega t + b_1(n) \cos \omega t \]

where

\[ b_0(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\text{as} \sin \omega t) \, dt \]

\[ a_1(n) = \frac{1}{\pi} \int_{0}^{2\pi} f(\text{as} \sin \omega t) \sin \omega t \, dt \]

\[ b_1(n) = \frac{1}{\pi} \int_{0}^{2\pi} f(\text{as} \sin \omega t) \cos \omega t \, dt \]

- Approximation made in neglecting higher frequency terms.
- Valid approximation if \( f(\cdot) \) is part of a larger system which attenuates higher frequencies.

\[ \approx b_0(n) + \sqrt{a_1(n)^2 + b_1(n)^2} \sin (\omega t + \varphi(n)) \]

\[ \varphi(n) = \tan^{-1} \frac{b_1(n)}{a_1(n)} \]
Define \( N(a) \) (the describing function) which is the transfer function between input to \( f \) and 1st harmonic in output:

\[
N(a) := \frac{\sqrt{a, (a)^2 + b, (a)^2}}{a} e^{j \phi(a)}
\]

\[
= \frac{a, (a)}{a} + j \frac{b, (a)}{a}
\]

In the above, we made the additional approximation that \( b_0(a) = 0 \) — this is valid for skew-symmetric \( f \), i.e.

\[
x = -f(-x)
\]

which is often a valid approximation since the kinds of nonlinearities often seen in engineering practice are:

RELAYS WITH DEADBAND.

RELAYS WITH Hysteresis.
EXAMPLES:

I. RELAY WITH DEADBAND.

Clearly, if \[ a < \varepsilon \], \( N(a) = 0 \).

Suppose \[ a \geq \varepsilon \]

\[ a, (a) = \frac{1}{\pi} \int_{0}^{2\pi} f(\sin wt) \sin wt \, d(wt) \]
\[ = \frac{1}{\pi} \cdot 4 \int_{0}^{\pi/2} \sin wt \, d(wt) \]
where \( \sin \phi = \varepsilon \).
\[ a_1(a) = \frac{4}{\pi} \cos \omega \frac{2\pi}{a} \sqrt{1 - \frac{\varepsilon^2}{a^2}} \]

\[ b_1(a) = \frac{1}{\pi} \int_0^{2\pi} f(\sin \omega t) \cos \omega t \, dt \]

\[ = 0 \quad \text{(all area cancels out over thus integral)} \quad \text{(Fig. ***)} \]

\[ a < \varepsilon, \quad N(a) = 0 \]

\[ a \geq \varepsilon, \quad N(a) = \frac{a_1(a)}{a} = \frac{4}{a\pi} \sqrt{1 - \frac{\varepsilon^2}{a^2}} \]

Simplification:

Since \( a_1(a), b_1(a) \) are always independent of \( \omega \):

Evaluate for \( \omega = 1 \)

\[ a_1(a) = \int_0^{2\pi} f(\sin t) \sin t \, dt / \pi \]

\[ b_1(a) = \int_0^{2\pi} f(\sin t) \cos t \, dt / \pi. \]
FACT 1. For SVSS FN $f$:

$$b_1(a) = 0$$

$$\Rightarrow N(a) = \frac{a_1(a)}{a} + j \frac{b_1(a)}{a} \in \mathbb{R} \quad \forall a \geq 0$$

Why?

$$b_1(a) = \frac{2\pi}{0} \int f(\text{asint}) \cos t \, dt$$

e.g.

$$\Rightarrow b_1(a) = \frac{1}{\pi} \int_0^{2\pi} f(\text{asint}) \cos t \, dt = 0.$$
**FACT 2** For DVSS function $f$:

For $a \geq \bar{a}$, because if $a < \bar{a}$ then the value of $f(asint)$ is not defined when $sint$ decreases after reaching its first maximum.

For $a \geq \bar{a}$:

$$a_1(a) = \frac{1}{\pi} \int_0^{2\pi} \frac{[f_1(asint) + f_2(asint)] \sin t}{2} dt$$

$$b_1(a) = \frac{2}{a\pi} \int_0^a [f_1(x) - f_2(x)] dx$$

$$= \left(-\frac{1}{a\pi}\right) \text{(area of loop)}$$

... makes evaluation of $b_1(a)$ easy
EXAMPLE II. RELAY WITH HYSTERESIS

\[ N(a) \text{ is only sensible for } a \geq \varepsilon \]

\[ a_1(a) = \frac{1}{\pi} \int_0^{2\pi} \frac{[f_1(a\sin t) + f_2(a\sin t)] \sin t \, dt}{2} \]

\[ \text{FACT 2} \]

but \( \frac{f_1(a\sin t) + f_2(a\sin t)}{2} \) is a relay with deadband!

\[ \Rightarrow a_1(a) = \frac{4}{a\pi} \sqrt{a^2 - \varepsilon^2} - \frac{4\varepsilon}{a\pi} \]

\[ \text{FACT 2} \]

\[ b_1(a) = (-\frac{1}{a\pi}) \left[ \text{area of loop} \right] \]

\[ = -\frac{4\varepsilon}{a\pi} \]

hence, for \( a \geq \varepsilon \),

\[ N(a) = \frac{a_1(a)}{a} + j \frac{b_1(a)}{a} \]

\[ = \frac{4}{a^2\pi} \sqrt{a^2 - \varepsilon^2} - j \frac{4\varepsilon}{a^2\pi} \]
EXAMPLE II (cont'd).

\[ \text{Im} \rightarrow \text{Re} \]

Increasing \( a \)

- \( a = \epsilon \)
- \( \frac{A}{\pi \epsilon} \)

- \( \frac{-1}{N(a)} \) - locus (semi circular)

EXAMPLE III

\[ \int f(x) = x^3 \]

\( f \) is SVSS \( \Rightarrow b_1(a) = 0 \)

\[ \text{FACT 1} \]

\[ f(\epsilon \sin t) = a^3 \sin^3 t = a^3 \left[ 3 \sin t - \sin 3t \right]/4 \]

Since \( \sin 3t = 3 \sin t - 4 \sin^3 t \)

Keeping only the 1st harmonic: \( \frac{3}{4} a^3 \sin t \)

\[ \Rightarrow a_1(a) = \frac{3}{4} a^3 \]

\[ \Rightarrow N(a) = \frac{3}{4} a^2 \]

\( -\frac{1}{N(a)} \) - locus

Increasing \( a \)